A Converse Lyapunov Theorem for p-dominant Switched Linear Systems

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***Abstract*— We study the path-complete p-contraction property for switched linear systems, which is a generalization of the notion of positive systems. We show on examples that this property is indeed useful for describing convergence properties, like p-dominance, that classical positivity cannot handle. We then provide a Converse Lyapunov Theorem, showing that, contrary to positivity, any p-dominant switched system must possess the path-complete p-contraction property with quadratic cones.**

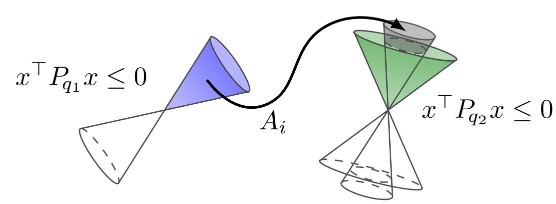
I. INTRODUCTION

Dynamical systems with a low-rank asymptotic behavior generally allow for a simplified analysis of their dynamics. For linear time-invariant (LTI) systems, this property, sometimes referred to as *model order reduction*, is well understood: a LTI system is amenable to model order reduction if it has p slow modes and $n-p$ fast modes (typically, $p \ll n$). By considering only the slow modes, we may reduce to the analysis of a p-dimensional system that inherits most of the asymptotic properties of the original system.

The seminal example of linear systems which have a low-dimensional asymptotic behavior are *positive systems*, that is, systems with a strictly invariant cone. By the Perron–Frobenius Theorem, positive systems have a single dominant eigenvector which is a 1-dimensional attractor for the system [1]. p-dominant linear systems extend this property to p-dimensional attractors by means of *quadratic p-cones*, that is, cones that can be described as the set of points x such that $x^\top Px \leq 0$ where P is a symmetric matrix with p negative eigenvalues and $n-p$ positive eigenvalues [2]. As in the case of positive systems, the existence of a strictly invariant quadratic p-cone for the linear system implies that the system has p dominant eigenvalues whose associated eigenspace is thus a p-dimensional attractor for the system.

In recent papers, the above concepts have been generalized to *switched linear systems*, that is, systems having several operating modes, each of them described by a linear system (or equivalently, linear systems with a time-dependent transition matrix $A(t)$ which can take only a finite number of different values for every t). For instance, the notion of positive system has been generalized to switched linear systems in two independent works: in [3, 4], under the name of "strictly invariant multicones"; and in [5], as "path-complete positivity". The first one uses *non-convex* cones and the second one uses *several* cones with contraction properties driven by the transitions in an automaton representing the system. Switched linear systems with strictly invariant multicones and path-positive switched linear systems enjoy similar asymptotic properties as positive linear systems.

Using ideas from path-positivity and path-complete Lyapunov theory [6], the recent paper [7] introduces a generalization of p-dominance for switched linear systems, under the name of path-complete p-dominance. Path-complete p-dominant switched linear systems are characterized by the existence of a family of quadratic p-cones which are contracted into each other following some rules driven by an automaton representing the system.



The works [5] and [7] focus on giving a geometric characterization of path-positive (resp. path-complete p-dominant) systems while showing that this gives a sufficient condition for the system to have a 1-dimensional (resp. p-dimensional) asymptotic behavior. One advantage of this geometric characterization is that it can be tested algorithmically thereby providing a *computable* sufficient criterion for a system to be asymptotically low-dimensional [5] (reps. [7]).

The aim of this paper is to further study the link between the "behavioral" characterization (p-dimensional asymptotic behavior) and the geometric characterization (existence of a contracting family of quadratic p-cones) for p-dominant switched linear systems. We already know form [7] that the geometric characterization is a sufficient condition for the behavioral one. In this paper, we prove the converse of this result: if a switched linear system has a p-dimensional asymptotic behavior, then there exists an automaton representing the system and a family of quadratic p-cones that are contracted into each other following some rules driven by this automaton. (The existence of such a family of cones can be seen as a Lyapunov criterion for p-dominance, hence the converse result is referred to as a *Converse Lyapunov Theorem* for p-dominance.)

Breaking with the terminology introduced in [7], we will use the term "p-dominant" to denote systems whose asymptotic behavior is p-dimensional while systems admitting a family of quadratic p-cones that are contracted into each other with respect to some automaton representing the system will be referred to as "path-complete p-contracting systems". The main result of this paper can then be reformulated as

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follows: a switched linear system system is p-dominant if and only if it is path-complete p-contracting (Theorem 2). As a corollary, we also deduce that the p-dominance property is robust to system perturbations (Corollary 3).

The paper is organized as follows: in Section II, we discuss the notions of p-dominance and path-complete p-contraction for LTI and switched linear systems. Then, we present the main result of the paper: the Converse Lyapunov Theorem for p-dominance. The proof of the Converse Lyapunov Theorem is provided in Section III. Finally, in Section IV, we illustrate the implications of the theorem on a numerical example.

II. p-DOMINANT SYSTEMS

A. LTI systems

A linear time-invariant (LTI) system

$$x(t+1) = Ax(t), \quad A \in \mathbb{R}^{n\times n}, \tag{1}$$

is p-dominant if the eigenvalues of A satisfy

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_p| > |\lambda_{p+1}| \geq \ldots \geq |\lambda_n|. \tag{2}$$

In this case, the state space can be decomposed as $\mathbb{R}^n = H \oplus V$ where H and V are the eigenspaces corresponding respectively to the first p and last $n-p$ eigenvalues of A (obtained, e.g., from the Jordan form). For almost every initial condition x_0, the component $x_{t,v}$ of $x(t)$ in V will become negligible compared to the component $x_{t,h}$ in H:

$$|A^t x_{0,v}| \leq C\lambda^t |A^t x_{0,h}| \qquad \forall t \geq 0 \tag{3}$$

(assume $x_{0,h} \neq 0$) for some $C \geq 0$ and $0 \leq \lambda < 1$.

Using the Main Inertia Theorem [8, §13.2], the property (2) can be formulated with a Linear Matrix Inequality (LMI) involving the matrix A and a symmetric matrix S of fixed inertia (the *inertia* of a symmetric matrix S is the triplet (i_-, i_0, i_+) where i_-, i_0 and i_+ are the number of negative, zero, and positive eigenvalues of S, respectively):

Proposition 1. A matrix $A \in \mathbb{R}^{n\times n}$ satisfies (2) if and only if there exist a symmetric matrix S with inertia $(p, 0, n-p)$ and a *rate* $\gamma > 0$ satisfying $A^\top SA - \gamma^2 S \prec 0$.

Given $S \in \mathbb{R}^{n\times n}$ symmetric with inertia $(p, 0, n-p)$, we define the *quadratic p-cone* $\mathcal{K}(S) = \{x \in \mathbb{R}^n : x^\top Sx \leq 0\}$. The degree p stands for the maximal dimension of a linear subspace contained in $\mathcal{K}(S)$. The geometric interpretation of Proposition 1 then reads as follows [7]: (1) is p-dominant if and only if there exists a quadratic p-cone $\mathcal{K}(S)$ satisfying

$$A\big(\mathcal{K}(S) \setminus \{0\}\big) \subseteq \operatorname{int} \mathcal{K}(S) \tag{4}$$

where $\operatorname{int} \mathcal{K}(S)$ denotes the interior of $\mathcal{K}(S)$.

B. Switched linear systems

We extend the notion of p-dominance to switched linear systems, that is, systems of the form

$$x(t+1) = A_{w(t)} x(t) \tag{5}$$

where w is a function from $\mathbb{Z}$ to $\Sigma = \{1, \ldots, N\}$ and $A_i \in \mathbb{R}^{n\times n}$ for every $i \in \Sigma$. The system is *constrained* if w is restricted to some subset $\mathcal{L} \subsetneq \Sigma^{\mathbb{Z}}$. In this case, $\mathcal{L}$ is called the *admissible language* and $\Sigma = \{1, \ldots, N\}$ is called the *alphabet*. Any $w \in \Sigma^{\mathbb{Z}}$ is called a *word* over Σ. Observe that a switched linear system is completely determined by the ordered pair $(\mathcal{M}, \mathcal{L})$ where $\mathcal{M} = \{A_i\}_{i\in\Sigma}$ is the indexed set of transition matrices of (5).

The definition of p-dominance (see Definition 1 below) for switched systems draws upon the behavioral characterization (3) of p-dominant LTI systems. The following notation will simplify the exposition: for $w \in \Sigma^{\mathbb{Z}}$ and $s < t$, let

$$A_w(s,t) = A_{w(t-1)} A_{w(t-2)} \cdots A_{w(s+1)} A_{w(s)},$$

and $A_w(s,t) = I_n$ if $s = t$. Thus, for all $w \in \mathcal{L}$ and $\bar{x} \in \mathbb{R}^n$, $x(t) = A_w(s,t)\bar{x}$ is a solution (aka. *trajectory* or *orbit*) for $t \geq s$ of (5) with input w and initial condition $x(s) = \bar{x}$.

Definition 1. A switched linear system $(\mathcal{M}, \mathcal{L})$ is called *p-dominant* if there exist $0 \leq \lambda < 1$ and $C \geq 1$ such that, for every $w \in \mathcal{L}$, there exist two collections of subspaces

$$\bar{H}_w = \{\ldots, H_{-1}, H_0, H_1, H_2, \ldots\},$$
$$\bar{V}_w = \{\ldots, V_{-1}, V_0, V_1, V_2, \ldots\}$$

where H_t and V_t are respectively p-dimensional and $(n-p)$-dimensional linear subspaces satisfying $\mathbb{R}^n = H_t \oplus V_t$ for every $t \in \mathbb{Z}$, and such that (i) for every $s \leq t$, $A_w(s,t)H_s = H_t$ (note the "equal") and $A_w(s,t)V_s \subseteq V_t$, and (ii) for every $s \leq t$, every $x_v \in V_s \setminus \{0\}$ and every $x_h \in H_s \setminus \{0\}$,

$$\frac{|A_w(s,t)x_v|}{|x_v|} \leq C\lambda^{t-s} \frac{|A_w(s,t)x_h|}{|x_h|}. \tag{6}$$

The pair $(\bar{H}_w, \bar{V}_w)$ in Definition 1 is called a *dominated invariant splitting* for w. The interpretation of p-dominance is that for every input w, the sequence of subspaces given by $\bar{H}_w$ defines a moving p-dimensional robust attractor for the system. More precisely, for every $w \in \mathcal{L}$ and almost every $x(0) \in \mathbb{R}^n$, the component of $x(t)$ in V_t will become negligible compared to the component of $x(t)$ in H_t as $t \to \infty$. That (6) must be satisfied for every $s \leq t$ (or, at least, for every $0 \leq s \leq t$) is required for the robustness of the attractor as demonstrated in Example 1 below.

Finally, in this paper, we have assumed that the input signal w is backward and forward infinite. The backward infinity assumption can be legitimated if we assume that the system we are considering is running in continuous-stream mode, i.e., we can assume that the system has started a long time ago. This is also the appropriate setting to study periodic input signals w. At the cost of adding very mild assumptions on the admissible language (e.g., $\mathcal{L}$ is a shift-invariant), there is no restriction in considering backward and forward infinite words instead of forward infinite words (which would seem more natural for the study of dynamical systems). However, for simplicity and conciseness of the presentation, we will focus on bi-infinite words in this paper.

Example 1. Define the 2×2 diagonal matrices

$$A_1 = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}.$$

Consider the switched linear system with $\mathcal{M} = \{A_1, A_2\}$ and $\mathcal{L} = \mathsf{X}_{21} := \{$all the words that do not contain 21 as a subword}. Hence, every sequences of matrices have one of the following forms: (a) $\ldots A_1 A_1 \ldots A_1 \ldots$, or (b) $\ldots A_1 A_1 \ldots A_1 A_2 \ldots A_2 A_2 \ldots$, or (c) $\ldots A_2 A_2 \ldots A_2 \ldots$.

In case (a), (6) is satisfied with $H_t = \mathbb{R} \times \{0\}$ and $V_t = \{0\} \times \mathbb{R}$ for every $t \in \mathbb{Z}$. In case (c), (6) is satisfied with $H_t = \{0\} \times \mathbb{R}$ and $V_t = \mathbb{R} \times \{0\}$ for every $t \in \mathbb{Z}$.

If we require that (6) must be satisfied only for $s = 0$ and $t \geq 0$, then case (b) satisfies (6) with $H_t = \mathbb{R} \times \{0\}$ and $V_t = \{0\} \times \mathbb{R}$ (resp. $H_t = \{0\} \times \mathbb{R}$ and $V_t = \mathbb{R} \times \{0\}$) if $w(0) = 1$ (resp. $w(0) = 2$). However, when $w(0) = 1$, the p-dimensional attractor $H_t = \mathbb{R} \times \{0\}$ will *not be robust*, as a small perturbation of the state of the system (from the moment that $w(t) = 2$) will cause the system to converge to $\{0\} \times \mathbb{R}$, and thus to diverge from $H_t = \mathbb{R} \times \{0\}$. $\triangleleft$

Figure 1 shows the behavior of a 2-dominant switched linear system whose trajectories are radially scaled to the unit sphere. The 2-dominant behavior of the system is captured by the convergence of all trajectories to a (time-varying) two-dimensional plane.

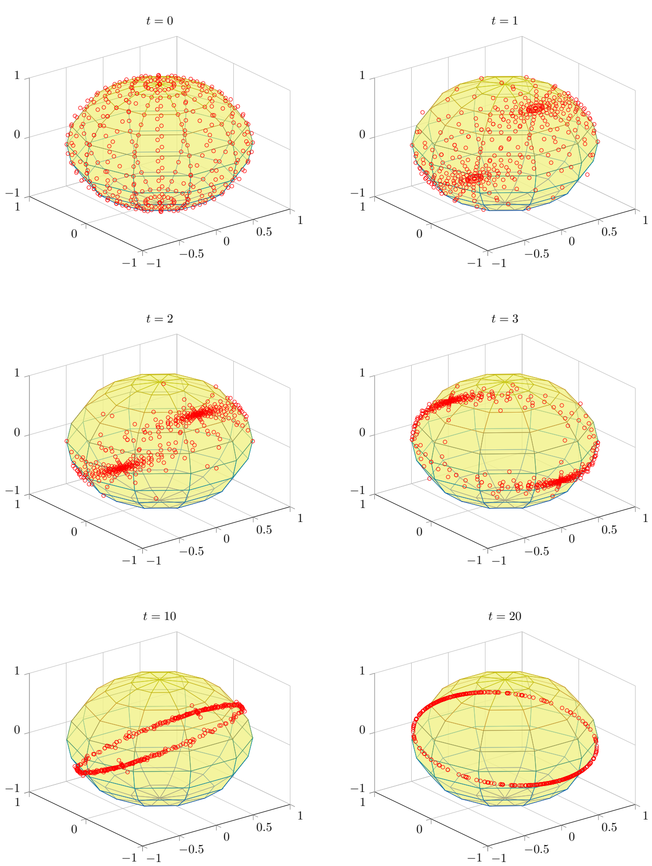


Fig. 1. Trajectories of a 2-dominant switched system from different initial conditions and for a fixed signal w. Each red dot represents the projection on the sphere of a trajectory $x(\cdot)$ at different times t.

C. Path-complete p-contracting switched linear systems

In the previous subsection, we have introduced the notion of p-dominance which characterizes the asymptotic behavior of switched linear systems. In this subsection, we describe another feature, called the *path-complete p-contraction* property, of switched linear systems that relies on the existence of strictly invariant sets (in particular quadratic p-cones) for the system. The main result of this paper is to show that the two notions coincide (Theorem 2 below).

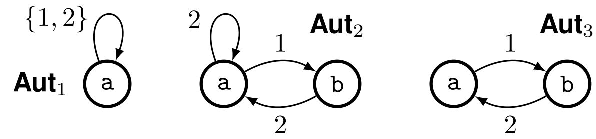


Fig. 2. Three automata with $\Sigma = \{1, 2\}$ and $Q = \{\mathtt{a}\}$ (for $\mathbf{Aut}_1$) or $Q = \{\mathtt{a}, \mathtt{b}\}$ (for $\mathbf{Aut}_2$ and $\mathbf{Aut}_3$). $\mathbf{Aut}_3$ accepts every words with a strict alternation of 1 and 2. $\mathbf{Aut}_2$ accepts every words that contains no consecutive 1's. $\mathbf{Aut}_1$ accepts every words on the alphabet $\{1, 2\}$. Indeed, every word admissible for the $\mathbf{Aut}_3$ is also admissible for $\mathbf{Aut}_1$ and $\mathbf{Aut}_2$.

Definition 2. A *finite-state automaton* (or *automaton* for short) $\mathbf{Aut}$ is a triplet (Q, Σ, δ) where Q is the (finite) set of *states*, $\Sigma = \{1, \ldots, N\}$ is the *alphabet* and $\delta \subseteq Q \times \Sigma \times Q$ is the set of *admissible transitions*. We will write $q_1 \xrightarrow{i} q_2 \in \delta$ if $(q_1, i, q_2) \in \delta$. A word $w \in \Sigma^{\mathbb{Z}}$ is *admissible* for $\mathbf{Aut}$ if there exists a bi-infinite sequence of states $\{q_t\}_{t \in \mathbb{Z}} \subseteq Q$ such that $q_t \xrightarrow{w(t)} q_{t+1} \in \delta$ for every $t \in \mathbb{Z}$. We say that the automaton $\mathbf{Aut}$ is *path-complete* for the language $\mathcal{L}$ if every word in $\mathcal{L}$ is admissible for $\mathbf{Aut}$.

See Figure 2 for an illustration.

Definition 3 (Path-complete p-contracting). Let $(\mathcal{M}, \mathcal{L})$ be a switched linear system.

a) We say that $\mathcal{M}$ is *p-contracting* with respect to the automaton $\mathbf{Aut} = (Q, \Sigma, \delta)$ if there exist (i) a set of symmetric matrices $\{S_q\}_{q \in Q} \subseteq \mathbb{R}^{n \times n}$ with uniform inertia $\mathrm{In}(S_q) = (p, 0, n - p)$ for every $q \in Q$, (ii) a set of *rates* $\{\gamma_d\}_{d \in \delta} \subseteq \mathbb{R}_{>0}$, and (iii) an $\varepsilon > 0$ such that for every transition $q_1 \xrightarrow{i} q_2 \in \delta$,
$$A_i^\top S_{q_2} A_i - \gamma_d^2 S_{q_1} \preceq -\varepsilon I. \qquad (7)$$

b) We say that $(\mathcal{M}, \mathcal{L})$ is *path-complete p-contracting* if there exists an automaton $\mathbf{Aut}$ that is path-complete for $\mathcal{L}$, and $\mathcal{M}$ is p-contracting with respect to $\mathbf{Aut}$.

Similarly to (4) for LTI systems, (7) expresses that the quadratic p-cone $\mathcal{K}(S_{q_1})$ is contracted into the quadratic p-cone $\mathcal{K}(S_{q_2})$ by the linear mapping A_i, i.e.,
$$(7) \iff A_i\big[\mathcal{K}(S_{q_1}) \setminus \{0\}\big] \subseteq \operatorname{int} \mathcal{K}(S_{q_2}).$$
See, e.g., [7, Proposition 2].

The main asset of the path-complete p-contraction criterion is that, for a given automaton, the p-contractivity of the system with respect to this automaton (Definition 3-a) can be tested algorithmically with LMI techniques [7, §4]. The next theorem states that the p-dominant switched linear systems are exactly the ones that are path-complete p-contracting. Hence, this gives an algorithmic framework to verify that a system is p-dominant.

Theorem 2. A switched linear system $(\mathcal{M}, \mathcal{L})$ is p-dominant if and only if it is path-complete p-contracting.

The "if" direction was proved in [7, Theorem 3]. The proof combines ideas from p-dominance for continuous-time systems [2, Theorem 1] and partial hyperbolicity [9]. The "if" direction can be seen as a Lyapunov theorem for p-dominance of switched linear systems as the p-dominance of the system is guaranteed by the decrease of the functions $V_i(x) = x^\top S_{q_i} x$. The difference with the classical Lyapunov theory is that we have several Lyapunov functions $V_i(x)$ and these functions are not positive-definite. The proof of the "only if" direction, called the *Converse Lyapunov Theorem* for p-dominance analysis, is provided in the next section.

From the Converse Lyapunov Theorem, we deduce that the p-dominance property is *robust to small perturbations of the system*. More precisely, a switched linear system $(\mathcal{M}', \mathcal{L}')$, $\mathcal{M}' = \{A'_1, \ldots, A'_N\}$, is said to be η-close to the switched linear system $(\mathcal{M}, \mathcal{L})$, $\mathcal{M} = \{A_1, \ldots, A_N\}$, if $\mathcal{L}' = \mathcal{L}$ and $\|A'_i - A_i\| \leq \eta$ for all $1 \leq i \leq N$. Then, we have:

Corollary 3. Let $(\mathcal{M}, \mathcal{L})$ be a p-dominant switched linear system. Then, there exists $\eta > 0$ such that every switched linear system $(\mathcal{M}', \mathcal{L}')$ that is η-close to $(\mathcal{M}, \mathcal{L})$ is also p-dominant.

Proof: Using Theorem 2, if $(\mathcal{M}, \mathcal{L})$ is p-dominant, there exist a path-complete automaton **Aut**, a family of symmetric matrices S_q with inertia $(p, 0, n-p)$, a set of rates $\gamma_d > 0$ and $\varepsilon > 0$ satisfying (7). Because the left-hand term of (7) is continuous with respect to A_i, there is $\eta > 0$ such that $(A'_i)^\top S_{q_2} (A'_i) - \gamma_d^2 S_{q_1} \preceq -\frac{\varepsilon}{2} I$ whenever $\|A'_i - A_i\| \leq \eta$. Hence, we get that $(\mathcal{M}', \mathcal{L}')$ is path-complete p-contracting and thus p-dominant by Theorem 2. ■

III. PROOF OF THEOREM 2

Part 1: The dominated invariant splitting

Proposition 4. Let $(\mathcal{M}, \mathcal{L})$ be a p-dominant switched linear system. Then, the dominated invariant splitting $(\bar{H}_w, \bar{V}_w)$ in Definition 1 is unique for every $w \in \mathcal{L}$.

The proof relies on the following observation:

Lemma 5. Let $(\mathcal{M}, \mathcal{L})$ and $(\bar{H}_w, \bar{V}_w)$ be as above. Then, for every $s \leq t$, $A_w(s,t)^{-1} V_t \subseteq V_s$ (where A^{-1} denotes the *preimage* by A).

Proof: Suppose $x \notin V_s$. Then, x has a nonzero component x_h in H_s, and thus, from $A_w(s,t) H_s = H_t$ (Definition 1), $A_w(s,t)x$ has a nonzero component in H_t (because H_s and H_t have the same dimension). Thus, $A_w(s,t)x \notin V_t$. ■

Proof of Proposition 4: Let $(\bar{H}_w, \bar{V}_w)$ and $(\bar{H}'_w, \bar{V}'_w)$ be two dominated invariant splittings for $w \in \mathcal{L}$, possibly with different constants $C \neq C'$ and $\lambda \neq \lambda'$ in (6).

First, suppose $V_s \neq V'_s$ for some $s \in \mathbb{Z}$. Then, there exists $x \in V_s$ such that $x = y' + z'$ with $y' \in H'_s \setminus \{0\}$ and $z' \in V'_s$. Hence, by (6),

$$\frac{|A_w(s,t)x|}{|A_w(s,t)y'|} \to 1 \qquad \text{as} \quad t \to \infty$$

Similarly, starting from some point $x' = y + z$ with $x' \in V'_s$, $y \in H_s \setminus \{0\}$ and $z \in V_s$, we find that

$$\frac{|A_w(s,t)x'|}{|A_w(s,t)y|} \to 1 \qquad \text{as} \quad t \to \infty$$

Combining the two results above, we obtain

$$\frac{|A_w(s,t)x|}{|A_w(s,t)y|} \frac{|A_w(s,t)x'|}{|A_w(s,t)y'|} \to 1 \qquad \text{as} \quad t \to \infty,$$

a contradiction with (6). Hence, $V_s = V'_s$ for all $s \in \mathbb{Z}$.

By considering orbits in backward time, and using the fact that $A_w(s,t)$ is bijective between H_s and H_t and Lemma 5, a similar argument can be used to prove that $H_t = H'_t$. This concludes the proof of Proposition 4. ■

In the proofs below, to avoid confusion, we will sometimes denote the subspaces defined by $(\bar{H}_w, \bar{V}_w)$—the unique dominated invariant splitting associated to w—by H_t^w and V_t^w (instead of simply H_t and V_t). The following lemma shows that the growth of $A_{w(s)} = A_w(s, s+1)$ on H_s^w cannot be arbitrarily small.

Lemma 6. Let $(\mathcal{M}, \mathcal{L})$ be a p-dominant switched linear system with dominated invariant splitting $(\bar{H}_w, \bar{V}_w)$. There exists $\eta > 0$ such that, for every $w \in \mathcal{L}$ and every $s \in \mathbb{Z}$,

$$|A_{w(s)}x| \geq \eta |x| \qquad \forall x \in H_s^w. \tag{8}$$

Proof: Step 1: For a contradiction, suppose that for every $n \in \mathbb{Z}_{>0}$, there exist $w_n \in \mathcal{L}$, $s_n \in \mathbb{Z}$ and $x_n \in H_{s_n}^{w_n}$ such that $|x_n| = 1$ and $|A_{w_n(s_n)} x_n| \leq 1/n$. From the uniqueness of $\bar{H}_w$ (Proposition 4), we may assume that $s_n = 0$: indeed, if w_n has dominated invariant splitting $(\bar{H}_{w_n}, \bar{V}_{w_n})$ and w'_n is the word w_n shifted by s_n symbols, i.e., $w'_n(t) = w_n(t + s_n)$, then $H_t^{w'_n} = H_{t+s_n}^{w_n}$, and thus, x_n satisfies the same properties as above with w'_n and $s_n = 0$.

Fix $T \leq 0$ such that $C\lambda^{-T} < \frac{1}{2}$. From the finiteness of Σ, we may assume (taking a subsequence if necessary) that there exists $w \in \mathcal{L}$ such that for every $n \in \mathbb{Z}_{>0}$, $w_n(t) = w(t)$ for $t = T, \ldots, 0$. Denote $A = A_w(T, 0) = A_{w_n}(T, 0)$ and $A' = A_w(T, 1) = A_{w_n}(T, 1) = A_{w_n(0)} A$ for simplicity.

For each n, let x'_n be the unique point in $H_T^{w_n}$ such that $x_n = A x'_n$. Because $\|A\|$ is bounded, $|x'_n|$ is bounded from below by some $c > 0$ for all $n > 0$.

Step 2: Define $x''_n = x'_n / |x'_n|$. Taking a subsequence if necessary, we assume that x''_n converges to some x with $|x| = 1$. Since $|A'x| = \lim_n |A' x''_n| \leq \lim_n c^{-1} |A_{w_n(0)} x_n| = 0$ by definition of x_n, we have that $x \in \ker A'$. From Lemma 5, we have that $\ker A' = A_{w_n}(T, 1)^{-1}\{0\} \subseteq V_T^{w_n}$. It is also clear that $x''_n \in H_T^{w_n}$ from its definition. From (6) and the facts that $x \in V_T^{w_n}$, $x''_n \in H_T^{w_n}$ and $C\lambda^{-T} < \frac{1}{2}$, we have that $|Ax| \leq \frac{1}{2} |A x''_n|$. Since $|Ax| = \lim_n |A x''_n|$, we finally get that $|A x''_n| \to 0$ as $n \to \infty$.

Step 3: Thus, by (6), we have that, for every $\varepsilon > 0$, there is an n such that $|Ay| \leq \varepsilon |y|$ for every $y \in V_T^{w_n}$. Since the dimension of $V_T^{w_n}$ is $n - p$ and ε is arbitrary, this implies that A has at least $n - p$ zero singular values. Then, $\operatorname{Im} A$ has dimension at most p, and thus, $H_0^{w_n} = \operatorname{Im} A$ independently of n, contradicting the assumption that for every n there is $x_n \in H_0^{w_n}$ such that $|x_n| = 1$ and $|A_{w_n(0)} x_n| \leq 1/n$. ■

Part 2: Building of the path-complete automaton

We will use projection matrices to describe the dominated invariant splittings $(\bar{H}_w, \bar{V}_w)$. Given a decomposition $\mathbb{R}^n = V_t^w \oplus H_t^w$, we define the matrix $P_t^w \in \mathbb{R}^{n \times n}$ as the projection on H_t^w parallel to V_t^w. (Note that P_t^w determines V_t^w and H_t^w completely since $\operatorname{Im} P_t^w = H_t^w$ and $\ker P_t^w = V_t^w$.) The invariance property translates as $A_w(s,t) \circ P_s^w = P_t^w \circ A_w(s,t)$, and the dimension condition as $\operatorname{rank} P_t^w = p$.

Proposition 7. Let $(\mathcal{M}, \mathcal{L})$ be a p-dominant switched linear system. There is $M \geq 0$ such that $\|P_t^w\| \leq M$ for every $w \in \mathcal{L}$ and every $t \in \mathbb{Z}$.

Proof: Fix $T \geq 0$ such that $C\lambda^T < \frac{1}{2}$ and define $M = 3\max\{\|A_w(t, t{+}T)\| : w \in \mathcal{L},\, t \in \mathbb{Z}\}/\eta^T$ where η is as in Lemma 6. We claim that $\|P_t^w\| \leq M$. Indeed, suppose for a contradiction there exists $x \in \mathbb{R}^n$ with $|x| = 1$ and $n_x := |P_t^w x| > M$. Define $y = P_t^w x/n_x$ and $z = (P_t^w x - x)/n_x$. Then $y \in H_t^w$ and $|y| = 1$; and $z \in V_t^w$ and $|z| \leq 1 + 1/n_x$. Let $A = A_w(t, t+T)$. From (6), $|Az|/|Ay| \leq \frac{1}{2}|z|/|y| \leq \frac{1}{2}(1 + 1/n_x) \leq \frac{2}{3}$ (because $M \geq 3$, thus $n_x \leq \frac{1}{3}$).

This implies that $|A(y - z)| \geq \frac{1}{3}|Ay| \geq \frac{1}{3}\eta^T$ (the latter coming from Lemma 6). This is a contradiction with $|y - z| = 1/n_x < 1/M$ and $\|A\| \leq \frac{1}{3}M\eta^T$, concluding the proof. ■

Proof of the "only if" direction of Theorem 2: To prove the "only if" direction, we suppose that we have a p-dominant switched system $(\mathcal{M}, \mathcal{L})$ with dominated invariant splittings described by the projection matrices P_t^w as above.

Step 1: Let $T \geq 1$ be such that $C\lambda^T < \frac{1}{4}$ and fix $0 < \zeta < \frac{3}{10}$. Let $\mathcal{P} = \{P_t^w : w \in \mathcal{L},\, t \in \mathbb{Z}\}$ and observe that $\mathcal{P}$ is a relatively compact subset of $\mathbb{R}^{n\times n}$ as it is a bounded subset (Proposition 7) of a finite-dimensional vector space. Hence, there is a finite set of rank-p projection operators $\{P_1, \ldots, P_\ell\}$ such that for every $P_t^w \in \mathcal{P}$, $\min_{1\leq k\leq \ell}\|P_t^w - P_k\| \leq \zeta$. In other words, $\{P_1, \ldots, P_\ell\}$ is a ζ-covering of $\mathcal{P}$.

Step 2: We build an automaton $\textbf{Aut}^* = (Q, \Sigma^T, \delta)$ and a set of symmetric matrices as follows: we let $Q = \{1, \ldots, \ell\}$ be the set of states of $\textbf{Aut}^*$, and for each $q \in Q$, we let

$$S_q = -P_q^\top P_q + (I - P_q)^\top (I - P_q) = I - P_q - P_q^\top. \quad (9)$$

Clearly, S_q is symmetric. Moreover, S_q is negative definite on $\operatorname{Im} P_q$ and positive definite on $\ker P_q$. Hence, $\operatorname{In}(S_q) = (p, 0, n-p)$ (by Courant–Fischer Theorem; see, e.g., [10]). The alphabet of $\textbf{Aut}^*$ is Σ^T, i.e., the set of words of length T over Σ: $w_0 w_1 \ldots w_{T-1} \in \Sigma^T$. Finally, we define the set $\delta \subseteq Q \times \Sigma^T \times Q$ of admissible transitions in $\textbf{Aut}^*$ as follows: for every $w \in \Sigma^T$, we let $q_1 \xrightarrow{w} q_2 \in \delta$ if and only if $A_w^\top S_{q_2} A_w \prec \kappa^2 S_{q_1}$ for some $\kappa > 0$ where $A_w = A_{w(T-1)} \cdots A_{w(0)}$.

Let $\mathcal{M}^T = \{A_w : w \in \Sigma^T\}$. By construction, $\mathcal{M}^T$ is p-contracting with respect to $\textbf{Aut}^*$. We will show that every word w in $\mathcal{L}$ can be read as the juxtaposition of length-T words obtain from a path in $\textbf{Aut}^*$: $w = \ldots\|u_{-1}\|u_0\|u_1\|\ldots$ where $\ldots \to u_{-1} \to u_0 \to u_1 \to \ldots$ is a path in $\textbf{Aut}^*$:

Step 3: To show this, let $w \in \mathcal{L}$ and $w = \ldots\|u_{-1}\|u_0\| u_1\|\ldots$ where $u_t = w(tT)\ldots w(tT + T - 1) \in \Sigma^T$. For every $t \in \mathbb{Z}$, let $q_t \in Q$ such that $\|P_{tT}^w - P_{q_t}\| \leq \zeta$. We claim that for every $t \in \mathbb{Z}$, $q_t \xrightarrow{u_t} q_{t+1} \in \delta$. To show this, we fix t and denote $A = A_{u_t} = A_w(tT, tT + T)$ for simplicity of notation. Let $x \in \mathbb{R}^n$, we will show that

$$x^\top S_{q_t} x < 0 \implies x^\top A^\top S_{q_{t+1}} A x < 0, \quad (10)$$

and the fact that $A_w^\top S_{q_{t+1}} A_{u_t} \prec \kappa^2 S_{q_t}$ for some $\kappa > 0$ will follow from the $\mathcal{S}$-Lemma (see, e.g., [11, §B.2]). Therefore, decompose $x = x_h + x_v$ where $x_h \in H_{tT}^w$ and $x_v \in V_{tT}^w$. Let $y = Ax$, $y_h = Ax_h$, and $y_v = Ax_v$. We use capital letters X_h, X_v, Y_h and Y_v to denote the norm of the related vectors. For instance, $X_h = |x_h|$.

Since $\|P_{tT}^w - P_{q_t}\| \leq \zeta$ and $\|P_{tT+T}^w - P_{q_{t+1}}\| \leq \zeta$, we have, from the definition (9) of S_q, that

$$x^\top S_{q_t} x < 0 \implies (1 + 2\zeta)X_h^2 > (1 - 2\zeta)X_v^2, \text{ and}$$
$$(1 - 2\zeta)Y_h^2 > (1 + 2\zeta)Y_v^2 \implies y^\top S_{q_{t+1}} y < 0.$$

We also have $Y_h/Y_v \geq 4X_h/X_v$ from (6) and the choice of T. Whence,

$$\frac{Y_h^2}{Y_v^2} \geq 16\frac{1 - 2\zeta}{1 + 2\zeta} > \frac{1 + 2\zeta}{1 - 2\zeta}$$

where we have used $\zeta < \frac{3}{10}$. This proves Step 3.

To conclude the proof, it remains to show that from $\textbf{Aut}^*$ we can build an automaton that is path-complete for $\mathcal{L}$ and for which $\mathcal{M}$ is p-contracting. This can be done by splitting each transition $q_1 \xrightarrow{u} q_2 \in \delta$ (with u a length-T-word) into T sub-transitions (one per symbol of u). We leave the details of the proof (which is not difficult) to the reader. ■

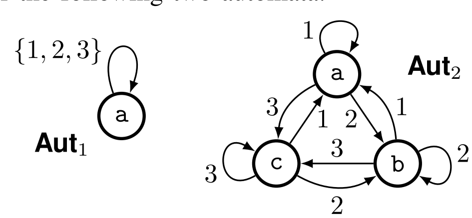
IV. NUMERICAL EXAMPLE

As mentioned in Section II, the main asset of the path-complete p-contraction criterion is that it can be efficiently computed for a given automaton. The Converse Lyapunov Theorem states that, provided that we consider large enough automata, the p-dominance of a switched linear system can always be verified using the path-complete p-contraction criterion. In this section, we illustrate on a numerical example the use of the criterion to prove the p-dominance of a switched linear system. The example also shows that, contrary to LTI systems, the p-dominance of switched linear systems cannot be reduced to the contraction of a single cone; showing thereby the necessity of resorting to non-trivial path-complete automata.

Define the 3×3 matrices

$$A_1 = \begin{bmatrix} 2 & \alpha & \alpha \\ 0 & 1-\alpha & 0 \\ 0 & 0 & 1-\alpha \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1-\alpha & 0 & 0 \\ \alpha & 2 & \alpha \\ 0 & 0 & 1-\alpha \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1-\alpha & 0 & 0 \\ 0 & 1-\alpha & 0 \\ \alpha & \alpha & 2 \end{bmatrix}$$

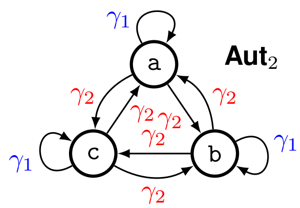
with $\alpha = 0.3$. We want to analyze the 1-dominance of the switched linear system $(\mathcal{M}, \mathcal{L})$ with $\mathcal{M} = \{A_1, A_2, A_3\}$ and $\mathcal{L} = \{1, 2, 3\}^{\mathbb{Z}}$ (unconstrained system). Therefore, we consider the following two automata:



which are both path-complete for $\mathcal{L}$.

$(\mathcal{M}, \mathcal{L})$ is not 1-contracting with respect to $\textbf{Aut}_1$ because, otherwise, this would mean there exists a quadratic 1-cone that is contracted by each A_i simultaneously. This can be proved impossible with a reasoning on the eigenspaces of the A_i's, and the fact that a cone contracted by A_i must contain only the dominant eigenvector A_i (see, e.g., [1]).

However, using the algorithmic framework described in [7], we can show that $(\mathcal{M}, \mathcal{L})$ is 1-contracting with respect to $\textbf{Aut}_2$. Indeed, if we use the following rates



$$\gamma_1 = 1.2, \quad \gamma_2 = 1.1,$$

then we find symmetric matrices $S_{\mathtt{a}}$, $S_{\mathtt{b}}$ and $S_{\mathtt{c}}$ with inertia $(1, 0, 2)$ satisfying (7) with respect to the above automaton. The quadratic 1-cones $\mathcal{K}(S_q) = \{x^\top S_q x \le 0\}$ associated to these matrices are represented in Figure 3.

The system is thus path-complete 1-contracting, and from Theorem 2, we conclude that it is 1-dominant. 1-dominance of the system asserts the existence of a family of 1-dimensional subspaces that will attract the trajectories of the system. For normalized trajectories, this implies the incremental stability property (trajectories converge to each other but not necessarily to a fixed point) for every switching signal $w \in \mathcal{L}$, as shown in Figure 4.

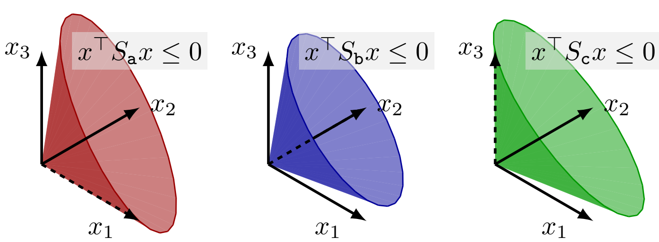


Fig. 3. Quadratic 1-cones $\mathcal{K}(S_q)$.

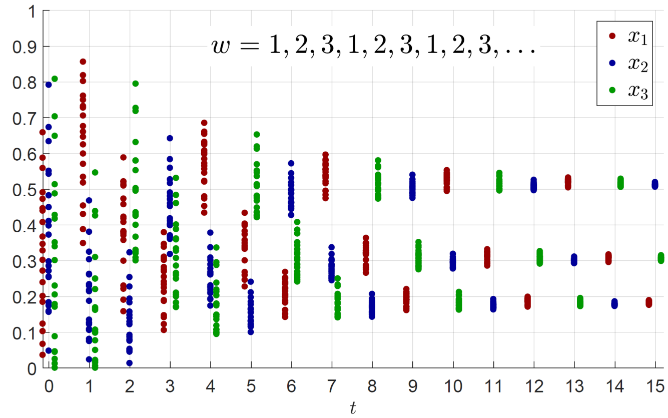


Fig. 4. Trajectories from different initial conditions $x = [x_1, x_2, x_3]^\top$, for a periodic signal $w = \{1, 2, 3, 1, 2, 3, \ldots\} \in \mathcal{L}$. All trajectories quickly converge to the same orbit, as predicted by 1-dominance.

V. CONCLUSIONS

The path-complete p-contraction property was introduced in [7] as a sufficient condition for a switched linear system to be p-dominant. p-dominance has received much attention in dynamical systems (aka. partially hyperbolic systems [9,12]) and control theory [13], as it allows to operate model order reduction on complex nonlinear systems. The advantage of the path-complete p-contraction criterion is that it can be efficiently verified with LMI techniques.

In the present paper, we have shown that the path-complete p-contraction property is also a necessary condition for p-dominance: if a switched linear system is p-dominant, then there always exist an automaton and a set of quadratic p-cones satisfying the p-contraction property. Moreover, we have given numerical examples showing that it is sometimes necessary to resort to non-trivial automata in order to prove the p-dominance of a particular system; thereby showing the importance of the "path-complete" part of the criterion.

As a further work, we plan to extend the path-complete p-contraction criterion to nonlinear systems. One way to do this would be to consider approximate *bisimulations* (aka. *abstractions*) of the system (see, e.g., [14]), and to build an abstract switched linear system from the bisimulation. Therefrom, drawing upon the results we obtained for switched linear systems, we plan to develop a similar algorithmic framework for the dominance analysis of general complex systems; with applications, e.g., in structural stability (aka. robustness) analysis, model order reduction, quantized control, etc.

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REFERENCES

[1] J. S. Vandergraft, "Spectral properties of matrices which have invariant cones," *SIAM Journal on Applied Mathematics*, vol. 16, no. 6, pp. 1208–1222, 1968.

[2] F. Forni and R. Sepulchre, "Differential dissipativity theory for dominance analysis," *IEEE Transactions on Automatic Control*, 2018.

[3] M. Brundu and M. Zennaro, "Invariant multicones for families of matrices," *Annali di Matematica Pura ed Applicata (1923-)*, pp. 1–44, 2018.

[4] J. Bochi and N. Gourmelon, "Some characterizations of domination," *Mathematische Zeitschrift*, vol. 263, no. 1, p. 221, 2009.

[5] F. Forni, R. M. Jungers, and R. Sepulchre, "Path-complete positivity of switching systems," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 4558–4563, 2017.

[6] D. Angeli, N. Athanasopoulos, R. M. Jungers, and M. Philippe, "Path-complete graphs and common lyapunov functions," in *Proceedings of the 20th International Conference on Hybrid Systems: Computation and Control*, pp. 81–90, ACM, 2017.

[7] G. O. Berger, F. Forni, and R. M. Jungers, "Path-complete p-dominant switching linear systems," in *2018 IEEE Conference on Decision and Control (CDC)*, pp. 6446–6451, IEEE, 2018.

[8] P. Lancaster and M. Tismenetsky, *The theory of matrices: with applications*. Elsevier, 1985.

[9] S. Newhouse, "Cone-fields, domination, and hyperbolicity," *Modern dynamical systems and applications*, pp. 419–432, 2004.

[10] R. A. Horn, R. A. Horn, and C. R. Johnson, *Matrix analysis*. Cambridge university press, 1990.

[11] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.

[12] B. Hasselblatt and Y. Pesin, "Partially hyperbolic dynamical systems," in *Handbook of dynamical systems*, vol. 1B, pp. 1–55, Elsevier, 2005.

[13] F. Colonius and W. Kliemann, *Dynamical systems and linear algebra*, vol. 158. American Mathematical Society, 2014.

[14] P. Tabuada, *Verification and control of hybrid systems: a symbolic approach*. Springer Science & Business Media, 2009.