p-dominant switched linear systems ⋆

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Abstract

This paper studies the asymptotic behavior of switched linear systems, beyond classical stability. We focus on systems having a low-dimensional asymptotic behavior, that is, systems whose trajectories converge to a common time-varying low-dimensional subspace. We introduce the concept of path-complete p-dominance for switched linear systems, which generalizes the approach of quadratic Lyapunov theory by replacing the contracting ellipsoids by families of quadratic cones whose contraction properties are dictated by an automaton. We show that path-complete p-dominant switched linear systems are exactly the ones that have a p-dimensional asymptotic behavior. Then, we describe an algorithm for the computation of the cones involved in the property of p-dominance. This allows us to provide an algorithmic framework for the analysis of switched linear systems with a low-dimensional asymptotic behavior.

Key words: Switched systems analysis, Switched linear systems, Linear matrix inequalities, Path-complete Lyapunov methods, Positive systems, Hyperbolic systems

1 Introduction

Positive systems, that is, linear systems that leave a convex pointed cone invariant, have been an important topic of research for some time now; see, e.g., Luenberger (1979), Berman et al. (1989), Kaczorek (2002) and Farina and Rinaldi (2000) for surveys. Indeed, positive systems appear naturally in a wide range of applications, such as economics, biology, Markov chains, opinion dynamics, etc. Moreover, the property of cone invariance provides significant information on the behavior of the system: namely, positive systems have a single dominant eigenvector (called Perron–Frobenius eigenvector) which is a 1-dimensional attractor for the system (Vandergraft, 1968). Consequently, positive systems allow for a simplified analysis and control of their dynamics; see, e.g., Luenberger (1979), Farina and Rinaldi (2000), Rantzer (2015) and references therein.

The concept of positive system has been generalized in several directions, such as: positive *time-varying* systems, i.e., linear time-varying systems leaving a convex pointed cone invariant (see, e.g., Parlett, 1970, and Pituk and Pötzsche, 2019); *monotone* systems, i.e., dynamical systems whose prolonged dynamics leaves a convex pointed cone invariant (see, e.g., Smith, 1995, Angeli and Sontag, 2003, and Hirsch and Smith, 2006); and more recently, *path-complete positive* systems (see Forni et al., 2017) and *differentially positive* systems (see Forni and Sepulchre, 2016) which further extend the property of cone invariance by moving from a single cone to a family of convex pointed cones. These generalizations enjoy similar properties as positive systems: in particular, their asymptotic behavior lies in a 1-dimensional object. This fundamental property has been used in a large number of contexts, e.g., for the analysis of Markov chains (Seneta, 1981), population dynamics (Parlett, 1970, Golubitsky et al., 1975), or communication networks (Shorten et al., 2006).

Recently, the concept of p-*dominance* was introduced by Forni and Sepulchre (2019) to generalize the approach of positivity to cones that are compatible with p-dimensional attractors. They show that continuous-time

⋆ G. Berger is a FRIA/FNRS fellow. R. Jungers is a FNRS honorary Research Associate. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 864017 – L2C. R. Jungers is also supported by the Innoviris Foundation and the FNRS (Chist-Era Druid-net). The material in this paper was partially presented at: (1) the 57th IEEE Conference on Decision and Control, December 17–19, 2018, Miami Beach, FL, USA. (2) the 18th European Control Conference, June 25–28, 2019, Naples, Italy.

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dynamical systems whose linearized dynamics leaves invariant a *quadratic p-cone* (that is, a cone described by a symmetric matrix with p negative eigenvalues and $n-p$ positive eigenvalues) have an asymptotic behavior that lies in a p-dimensional object. In this sense, the theory of p-dominance connects with the theory of *partial hyperbolicity* and *exponential dichotomy* (see, e.g., Brin and Pesin, 1974, and Barreira and Valls, 2008), dealing with systems whose linearized dynamics present an exponential separation between a p-dimensional dominant component and a complementary transient component. As for p-dominant systems, partially hyperbolic systems allow for a simplified analysis of their dynamics; see, e.g., Brin and Pesin (1974), Hirsch et al. (1977), Barreira and Valls (2008) and Pesin (2004).

In this work, we focus on discrete-time *switched linear systems* (SLSs). These are systems described by a finite set of linear modes among which the system can switch in time. As a paradigmatic class of cyber-physical systems and hybrid systems, SLSs have attracted much attention from the control community in recent years; see, e.g., Liberzon (2003) and Lin and Antsaklis (2009) for introductions. A large part of these works focuses on the question of stability, which already turns out to be extremely challenging (Tsitsiklis and Blondel, 1997). However, many complex systems encountered in applications are in fact not stable with respect to a single fixed point but nevertheless present a low-dimensional asymptotic behavior. The aim of this paper is to provide a computational framework for the analysis of such discrete-time SLSs having a p-dimensional asymptotic behavior, that is, whose trajectories converge to a time-varying p-dimensional subspace.

Our approach combines ideas from p-dominance analysis, discussed above, and from *path-complete Lyapunov theory*, introduced in the context of stability analysis of switched systems (see, e.g., Ahmadi et al., 2014, and Angeli et al., 2017). First, p-dominance is extended to *path-complete* p-dominance, by moving from a single quadratic p-cone to a family of quadratic p-cones whose invariance properties are driven by an automaton capturing the admissible switching sequences of the system (called a *path-complete* automaton). The goal is to increase the expressiveness of p-dominance analysis while preserving the feature of a p-dimensional asymptotic behavior: indeed, similarly to the case of stability analysis (for which quadratic Lyapunov functions are known to be conservative), p-dominance with respect to a single quadratic p-cone does not allow to capture all SLSs with a p-dimensional asymptotic behavior (see Example 16). The use of a path-complete automaton allows to alleviate this conservatism: in particular, we show that path-complete p-dominance is a necessary and sufficient condition for having a p-dimensional asymptotic behavior.

Secondly, we show that the property of path-complete p-dominance can be verified algorithmically. The use of quadratic p-cones allows to encode the invariance relations as the feasibility of a set of matrix inequalities. This property has been extensively used in the context of positivity and p-dominance analysis (see, e.g., Hildebrand, 2007, Grussler and Rantzer, 2014, Forni and Sepulchre, 2019), leading to efficient methods, based on conic optimization, for the computation of a single invariant quadratic p-cone. Thriving on these results, we provide an algorithm for the computation of families of quadratic p-cones whose invariance properties are driven by an automaton. Combined with the non-conservatism of path-complete p-dominance, this results in a tractable computational framework for the analysis of SLSs that have a p-dimensional asymptotic behavior.

Comments on earlier works. A preliminary discussion of the results presented in this paper has been reported in the conference papers: Berger et al. (2018) and Berger and Jungers (2019). The present work completes and improves these preliminary results in two ways:

1. Focus on the algorithmic aspects: The algorithmic aspects were absent from Berger and Jungers (2019), and only suggested without proofs in Berger et al. (2018). In this paper, we provide a thorough description and analysis of the computational framework for the verification of path-complete p-dominance, and we provide several examples of application, illustrating the practical applicability of the framework.

2. Improved presentation of the results and connections with other works: The presentation of the main result, linking the property of path-complete p-dominance with the property of having a p-dimensional asymptotic behavior, was initially split across Berger et al. (2018) and Berger and Jungers (2019). We improve the presentation of the result and we simplify and shorten its proof by unifying the notation and removing redundancies with the analysis of the algorithmic framework. We also provide several proofs that were not present in the conference papers; this is the case for instance for the last part of the proof of Theorem 7, for the proofs related to the algorithmic aspects. We also add several examples and figures illustrating the main concepts. Finally, we discuss the connections and comparisons of our work with other works in the literature; see Subsection 2.3.

Outline. The paper is organized as follows. The main concepts related to path-complete p-dominance of SLSs and the characterization of their asymptotic behavior are presented in Section 2. The algorithm for the verification of path-complete p-dominance is described and discussed in Section 3. Finally, numerical examples and examples of applications are presented in Section 4.

All proofs can be found in the appendix.

Notation. For vectors, $\|\cdot\|$ denotes the Euclidean norm, and for matrices, it denotes the spectral matrix norm.

The set of real $n \times n$ symmetric matrices is denoted by $\mathbb{S}^{n\times n}$. For $P, Q \in \mathbb{S}^{n\times n}$, we write $P \succ Q$ (resp. $P \succeq Q$) if $P - Q$ is positive definite (resp. positive semidefinite). A matrix $P \in \mathbb{S}^{n\times n}$ is said to have *inertia* $(k, 0, n-k)$ if it has k negative (< 0) eigenvalues and $n-k$ positive (> 0) eigenvalues; the set of all matrices $P \in \mathbb{S}^{n\times n}$ with inertia $(k, 0, n-k)$ is denoted by $\mathbb{S}^{n\times n}_k$. For $\mathcal{S} \subseteq \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, $A\mathcal{S} = A(\mathcal{S})$ denotes the image of $\mathcal{S}$ by A.

2 p-dominant switched linear systems

2.1 p-dominant switched linear systems

We consider *switched linear systems* (SLSs), that is, systems of the form

$$x(t+1) = A_{\sigma(t)}x(t), \quad t \in \mathbb{N}, \tag{1}$$

where $\sigma(t) \in \Sigma \coloneqq \{1, \dots, m\}$ and $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \Sigma$. The function $\sigma : \mathbb{N} \to \Sigma$ is called the *switching signal* (or *s.s.* for short) of the system, and it specifies which *mode* is used by the system at time t.

A usual way to represent the set of switching signals of an SLS is by using a *finite-state automaton*:

Definition 1.

- A *finite-state automaton* (or *automaton* for short) **Aut** is a triplet (Q, Σ, δ) where Q is the finite set of states, $\Sigma = \{1, \dots, m\}$ is the alphabet, and $\delta \subseteq Q \times \Sigma \times Q$ is the set of admissible transitions.
- For a transition $d = (q_1, i, q_2) \in \delta$, we denote its source q_1 by $\mathsf{s}(d)$, its target q_2 by $\mathsf{t}(d)$, and its label i by $\mathsf{i}(d)$. A *path* in **Aut** is any sequence $(d_t)_{t=0}^{N-1} \in \delta^N$ (where N can be infinite) such that $\mathsf{t}(d_t) = \mathsf{s}(d_{t+1})$ for all $t \in \{0, \dots, N-2\}$.
- A s.s. $\sigma \in \Sigma^{\mathbb{N}}$ is *admissible* for **Aut** if there exists an infinite path $(d_t)_{t=0}^{\infty}$ in **Aut** such that $\sigma(t) = \mathsf{i}(d_t)$ for every $t \in \mathbb{N}$. **Aut** is *path-complete* for the SLS (1) if every s.s. in $\Sigma^{\mathbb{N}}$ is admissible for **Aut**.

The notion of path-complete automaton is illustrated in Figure 1.

Given an automaton $\textbf{Aut} = (Q, \Sigma, \delta)$, we let $\{\gamma_d\}_{d\in\delta}$ be a set of positive *rates* (one per transition of the automaton). The property of path-complete p-dominance, introduced below, extends the approach of cone invariance (used in the analysis of positive and p-dominant systems; see Section 1) by considering a *set* of quadratic p-cones whose contraction properties are *driven by an automaton.* The quadratic p-cones are represented by symmetric matrices P_q with fixed inertia and the contraction properties are captured by matrix inequalities driven by the automaton and the set of rates $\{\gamma_d\}_{d\in\delta}$.

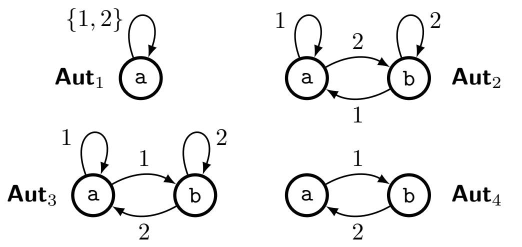

Fig. 1. Three automata with $\Sigma = \{1, 2\}$, and $Q = \{\mathsf{a}\}$ (for $\textbf{Aut}_1$) or $Q = \{\mathsf{a}, \mathsf{b}\}$ (for $\textbf{Aut}_2$, $\textbf{Aut}_3$ and $\textbf{Aut}_4$). The transitions are represented by the edges (i.e., $q_1 \xrightarrow{i} q_2$ if and only if $(q_1, i, q_2) \in \delta$). $\textbf{Aut}_1$, $\textbf{Aut}_2$ and $\textbf{Aut}_3$ are path-complete for the SLSs with set of modes $\Sigma = \{1, 2\}$, while $\textbf{Aut}_4$ is not.

Definition 2. Let $\textbf{Aut} = (Q, \Sigma, \delta)$ be an automaton together with a set of rates $\{\gamma_d\}_{d\in\delta} \subseteq \mathbb{R}_{>0}$. System (1) is said to be *p-dominant* with respect to **Aut** and $\{\gamma_d\}_{d\in\delta}$ if there is a set of matrices $\{P_q\}_{q\in Q} \subseteq \mathbb{S}^{n\times n}_p$ (i.e., all P_q have inertia $(p, 0, n-p)$) such that for every $d \in \delta$,

$$A_{\mathsf{i}(d)}^\top P_{\mathsf{t}(d)} A_{\mathsf{i}(d)} - \gamma_d^2 P_{\mathsf{s}(d)} \prec 0, \tag{2}$$

where $\mathsf{s}(d), \mathsf{t}(d), \mathsf{i}(d)$ are as in Definition 1.

Remark 3. For a given automaton and a given set of rates, there is at most one value of p for which the system is p-dominant; see also Proposition 14-(i). However, depending on the automaton and the set of rates, the system can be p-dominant for different values of p. ◁

Letting $V_q(x) = x^\top P_q x$, $q \in Q$, the dissipation inequalities (2) imply that there is $\varepsilon > 0$ such that for every trajectory $x(\cdot)$ of (1) with s.s. $\sigma \in \Sigma^{\mathbb{N}}$, $V_{\mathsf{s}(d_{t+1})}(x(t+1)) \leq \gamma_{d_t}^2 V_{\mathsf{s}(d_t)}(x(t)) - \varepsilon\|x(t)\|^2$, where $(d_t)_{t=0}^{\infty}$ is a path in **Aut** satisfying $\sigma(t) = \mathsf{i}(d_t)$ for every $t \in \mathbb{N}$. This implies that the family of *quadratic p-cones* defined by

$$\mathcal{K}(P_q) = \{x \in \mathbb{R}^n : V_q(x) \leq 0\}, \quad q \in Q,$$

is *contracted by the system*, in the sense that

$$A_{\sigma(t)}(\mathcal{K}(P_{\mathsf{s}(d_t)}) \setminus \{0\}) \subseteq \operatorname{int} \mathcal{K}(P_{\mathsf{s}(d_{t+1})}) \quad \forall\, t \in \mathbb{N}. \tag{3}$$

The example below illustrates the concept of p-dominant SLSs, and the contraction property (3).

Example 4. Consider System (1) with $\Sigma = \{1, 2\}$,

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1-\alpha & \alpha \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} \alpha & \alpha - 1 \\ 0 & 1 \end{bmatrix},$$

and $\alpha = 0.1$; which may occur for instance in the modeling of opinion dynamics with antagonistic interactions and switching topologies (Meng et al., 2016). This system is 1-dominant with respect to the automaton $\textbf{Aut}_2$ presented in Figure 1 and with the set of rates $\{\gamma_d\}_{d\in\delta}$

defined by $\gamma_d = 0.32$ for all $d \in \delta$,[1] meaning that there are matrices $P_{\mathsf{a}}, P_{\mathsf{b}} \in \mathbb{S}_1^{2\times 2}$ satisfying (2) with $\mathbf{Aut}_2$ and $\{\gamma_d\}_{d\in\delta}$. The quadratic 1-cones associated to P_{a} and P_{b} are represented in Figure 2. We observe that the cones satisfy the contraction property (3). $\triangleleft$

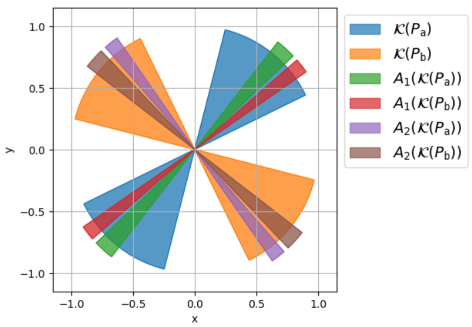

Fig. 2. Quadratic 1-cones $\mathcal{K}(P_{\mathsf{a}})$ and $\mathcal{K}(P_{\mathsf{b}})$ and their images by A_1 and A_2 (see Example 4).

2.1.1 *The case of LTI systems*

For an LTI system $x(t+1) = Ax(t)$, the property of p-dominance reduces to the feasibility of the matrix inequality $A^\top PA - \gamma^2 P \prec 0$ for some rate $\gamma > 0$ and some matrix $P \in \mathbb{S}_p^{n\times n}$. It is well known that this inequality implies that A has p eigenvalues with modulus $|\lambda_i| > \gamma$, and $n-p$ eigenvalues with modulus $|\lambda_i| < \gamma$; see, e.g., Theorem 20 (from Lancaster and Tismenetsky, 1985) in Appendix A. In this case, the eigenvalue decomposition of A implies that there is a splitting of the state space $\mathbb{R}^n = E \oplus F$, where E is a subspace with dimension $n-p$ satisfying $AE \subseteq E$ and F is a subspace with dimension p satisfying $AF = F$. Furthermore, there are constants $C \geq 1$ and $\mu \in (0,1)$ such that for any pair of trajectories, $x(\cdot)$ and $y(\cdot)$, of the system, with $x(0) \in E$ and $y(0) \in F \setminus \{0\}$, it holds that

$$\frac{\|x(t)\|}{\|y(t)\|} \leq \frac{\|x(0)\|}{\|y(0)\|} C\mu^t \quad \forall t \in \mathbb{N}. \tag{4}$$

The pair (E, F) is called a *dominated splitting* as it ensures a decomposition into p *dominant* modes and $n-p$ *transient* modes of the system. Another name for this property is that there is an *exponential dichotomy* (Barreira and Valls, 2008) at the equilibrium point 0.

Remark 5. For early references on the geometric characterization, see, e.g., Stern and Wolkowicz (1991) where it is shown that an LTI system admits a pointed invariant ellipsoidal cone if and only if it has a positive eigenvalue strictly larger in modulus than any other eigenvalue. Important classes of LTI systems satisfying the eigenvalue separation property of p-dominance include *relaxation systems* (see, e.g., Willems, 1976, and Pates et al., 2019), and *totally positive systems* (see, e.g., Margaliot and Sontag, 2019, Grussler and Sepulchre, 2020, and Grussler et al., 2021); indeed, for these systems, it holds that $\lambda_1 > \lambda_2 > \ldots > \lambda_n \geq 0$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A (see, e.g., Willems, 1976, Theorem 4, and Margaliot and Sontag, 2019, Theorem 1).

2.2 *Asymptotic behavior of p-dominant SLSs*

In this subsection, we show that p-dominant SLSs inherit the asymptotic properties of p-dominant LTI systems, in the sense that their asymptotic behavior is p-dimensional (a property formalized with a condition similar to (4)). The difference with the LTI case is that the p-dimensional subspace attractor is not fixed anymore, but may vary with time. To formalize this, we first introduce the notion of time-varying splitting:

Definition 6. A *time-varying splitting* (or *splitting* for short) of $\mathbb{R}^n$ is a pair $(\mathcal{E}, \mathcal{F})$ consisting of two sequences of linear subspaces $\mathcal{E} = (E_t)_{t=0}^\infty$ and $\mathcal{F} = (F_t)_{t=0}^\infty$ satisfying $\mathbb{R}^n = E_t \oplus F_t$ for all $t \in \mathbb{N}$.
We say that $(\mathcal{E}, \mathcal{F})$ is a *p-splitting* if each F_t has dimension p ($\Leftrightarrow$ each E_t has dimension $n-p$).

The following theorem is the first main result of this paper. It generalizes (4) to p-dominant SLSs, and also states the converse result, i.e., that any SLSs satisfying a condition similar to (4) must be p-dominant.

Theorem 7. *Consider System* (1). *The following are equivalent:*

- *(a) There is an automaton* $\mathbf{Aut} = (Q, \Sigma, \delta)$ *that is path-complete for* (1) *and a set of rates* $\{\gamma_d\}_{d\in\delta} \subseteq \mathbb{R}_{>0}$ *such that* (1) *is p-dominant with respect to* $\mathbf{Aut}$ *and* $\{\gamma_d\}_{d\in\delta}$.
- *(b) For every s.s.* $\sigma \in \Sigma^{\mathbb{N}}$, *there is a p-splitting* $(\mathcal{E}, \mathcal{F})$ *satisfying (i)* $A_{\sigma(t)} E_t \subseteq E_{t+1}$ *and* $A_{\sigma(t)} F_t = F_{t+1}$ *for all* $t \in \mathbb{N}$, *and (ii) for every* $s \in \mathbb{N}$ *and every pair of trajectories,* $x(\cdot)$ *and* $y(\cdot)$, *of* (1) *with s.s.* σ *and with* $x(s) \in E_s$ *and* $y(s) \in F_s \setminus \{0\}$, *it holds that*

$$\frac{\|x(t)\|}{\|y(t)\|} \leq \frac{\|x(s)\|}{\|y(s)\|} C\mu^{t-s} \quad \forall s, t \in \mathbb{N},\ t \geq s, \tag{5}$$

for some $C \geq 1$ *and* $\mu \in (0,1)$, *independent of* σ.

The pair $(\mathcal{E}, \mathcal{F})$ in Theorem 7-(b) is called a *dominated invariant splitting* for (1) with s.s. σ. The interpretation of (5) is that the sequence of subspaces given by $\mathcal{F}$ defines a robust time-varying p-dimensional attractor for the system. More precisely, for every trajectory of the system, the component of $x(t)$ in E_t will become negligible compared to the component of $x(t)$ in F_t as $t \to \infty$.

[1] The selection of the value of the rates will be discussed in Example 15, after we have presented a set of constraints that must be satisfied by the set of rates (Subsection 3.2). The verification of p-dominance was achieved by using the algorithm described in Subsection 3.1.

The fact that (5) holds for every $s \in \mathbb{N}$ (and not only for $s = 0$) ensures the robustness of the attractor; see also Berger and Jungers (2019, Example 1).

An SLS that satisfies Theorem 7-(a) will be said to be *path-complete* p*-dominant*. Theorem 7 can then be reformulated as follows. *An SLS is path-complete* p*-dominant if and only if it has a robust time-varying* p*-dimensional subspace attractor for every switching signal.* This result generalizes the one for LTI systems (see Subsubsection 2.1.1) to SLSs. However, by contrast to the LTI case, the time-varying nature of the system implies that the dominated invariant splitting may depend on time and on the switching signal. Another difference with the LTI case is that, while the reverse implication (b) $\Rightarrow$ (a) also holds for SLSs (see Theorem 7), the automaton required in (a) may be non-trivial. Indeed, in the LTI case, it is always sufficient to take for **Aut** the trivial automaton with one node and one transition, for (a) to be verified; however, in the case of SLSs, (a) may require a nontrivial automaton (i.e., with more than one node); this phenomenon is illustrated in Example 16 (Section 4).

The dominance property (5) is illustrated in Figure 3, which shows the behavior of the 1-dominant SLS of Example 4 and of a 2-dominant SLS [2]. The 1-dominant behavior of the first system is captured by the convergence of the normalized trajectories to two opposite "attracting trajectories" when $t \to \infty$ and for any s.s. $\sigma \in \{1, 2\}^{\mathbb{N}}$. The 2-dominant behavior of the second system is captured by the convergence of the normalized trajectories to a time-varying 2-dimensional plane, F_t, when $t \to \infty$. Unlike stable SLSs, whose trajectories all converge to a unique equilibrium, p-dominant SLSs allow for richer behaviors.

A straightforward consequence of the equivalence of (a) and (b) in Theorem 7 is that the existence of a dominated invariant p-splitting $(\mathcal{E}, \mathcal{F})$ is *robust to small perturbations of the system*. The robustness property is instrumental for numerical analysis, and also shows that the property of having a low-dimensional dominant behavior occurs with nonzero probability for SLSs.

Corollary 8. *Property (b) in Theorem 7 is robust to small perturbations of the matrices* $\{A_i\}_{i\in\Sigma}$.

Proof. Indeed, Property (a) is clearly robust to system perturbations, as for any small enough perturbation of the matrices $\{A_i\}_{i\in\Sigma}$, the dissipation inequalities (2) will still be satisfied. Hence, from the equivalence of (a) and (b), we get the desired result. $\square$

[2] The SLS is defined by the matrices $\begin{bmatrix} 1 & 0.5 & 0 \\ \alpha & 0.75 & 0.5 \\ -0.5 & 0 & 1.0 \end{bmatrix}$, where $\alpha \in \{-1, -0.8, -0.6, \ldots, 0\}$. This system can be shown to be 2-dominant using the algorithm presented in Section 3 (due to space limitation, the details are omitted).

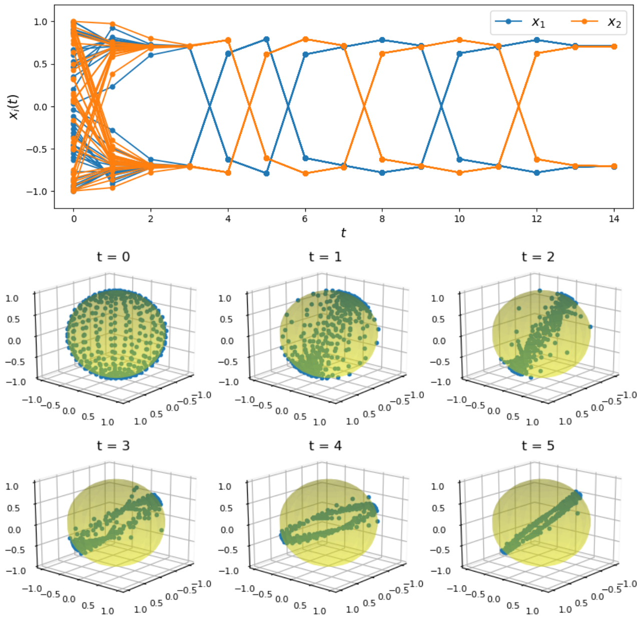

Fig. 3. *Top*: Normalized trajectories of the SLS from Example 4, starting from different initial conditions and for a random s.s. σ. *Bottom*: Normalized trajectories of a 2-dominant SLS starting from different initial conditions and for a random s.s. σ. Each dot represents the projection on the sphere of a trajectory $x(\cdot)$ at times $t = 0, 1, \ldots, 5$.

An interesting situation is when the system has a *stable transient behavior*. This means that the system converges to zero on the dominated component of the splitting (i.e., $\mathcal{E}$), so that the asymptotic behavior of the system is dictated by the dominant component (i.e., $\mathcal{F}$) only. In order to characterize SLSs with such a property, we introduce the notion of *cycle-stable* automaton; see also the *maximum cycle mean problem* in graph theory (Karp, 1978) and applications in switched systems analysis (Ahmadi and Parrilo, 2012).

Definition 9. Given an automaton $\mathbf{Aut} = (Q, \Sigma, \delta)$ together with a set of rates $\{\gamma_d\}_{d\in\delta} \subseteq \mathbb{R}_{>0}$, we say that **Aut** is *cycle-stable* with respect to $\{\gamma_d\}_{d\in\delta}$ if every cycle [3] $(d_t)_{t=0}^{N-1}$ in **Aut** satisfies $\gamma_{d_0} \ldots \gamma_{d_{N-1}} \le 1$.

We obtain the following characterization of SLSs with stable dynamics on the dominated component $\mathcal{E}$:

Theorem 10. *Consider System* (1)*. The following are equivalent:*

(a) *The system satisfies Property (a) in Theorem 7 and* **Aut** *is cycle-stable with respect to* $\{\gamma_d\}_{d\in\delta}$.
(b) *The system satisfies Property (b) in Theorem 7 and there are constants* $D \ge 1$ *and* $\rho \in (0, 1)$ *such that for every s.s.* $\sigma \in \Sigma^{\mathbb{N}}$, $s \in \mathbb{N}$ *and every trajectory*

[3] A cycle is a path $(d_t)_{t=0}^{N-1} \in \delta^N$ in **Aut** such that $\mathsf{t}(d_{N-1}) = \mathsf{s}(d_0)$ and $\mathsf{s}(d_s) \ne \mathsf{s}(d_t)$ for all $s \ne t$.

$x(\cdot)$ of (1) with s.s. σ and with $x(s) \in E_s$, it holds that

$$\|x(t)\| \leq \|x(s)\| D\rho^{t-s} \quad \forall\, s,t \in \mathbb{N},\, t \geq s.$$

Summarizing, in this section, we introduced the concept of path-complete p-dominance for SLSs, and showed that this concept was key for the theoretical analysis of SLSs with a low-dimensional asymptotic behavior, a property made precise thanks to the notion of dominated splitting; see Theorems 7 and 10. In Section 3, we will address the question of algorithmic verification of the property of path-complete p-dominance. Before this, in the next subsection, we discuss the connections of our approach with other works in the literature.

2.3 Discussion and connections with the literature

Our work connects with several other concepts in control and system theory. For instance, the use of a family of quadratic forms whose decay properties are dictated by an automaton is inspired from *path-complete Lyapunov functions* introduced in the context of stability analysis of switched systems (see, e.g., Ahmadi et al., 2014, and Angeli et al., 2017), and from *path-complete positivity* (see Forni et al., 2017) which extends the property of positivity by moving from a single contracting cone to a family of convex cones whose contraction properties are driven by an automaton.

Another important concept in our analysis is the one of *dominated splitting*, which was first introduced by Mañé (1987) in the context of *partial hyperbolicity* and *exponential dichotomy* theory (a generalization of the celebrated works of Smale and Anosov on the horseshoe map; see, e.g., Brin and Pesin, 1974, and Barreira and Valls, 2008). Dominated splittings also received attention in the study of some particular cases of SLSs; see, e.g., Avila et al. (2010), Bochi and Gourmelon (2009), Brundu and Zennaro (2019) and Barreira and Valls (2009). An important tool in these works is the notion of invariant *multicones*. In fact, the proof of our converse Lyapunov theorem for p-dominance is partially grounded in the proof of Bochi and Gourmelon (2009, Theorem B), which shows that an SLS with *invertible* matrices admits a dominated splitting for every s.s. if and only if it admits a contracting multicone. Our work extends this result to SLSs involving singular matrices and to families of quadratic p-cones whose contraction properties are dictated by an automaton. Another difference with these references is that little attention is given to the algorithmic decidability of the geometric property, whereas our approach is meant to be translated into a practical algorithm for the computation of the quadratic p-cones, as explained in the next section.

3 Algorithm for the verification of p-dominance of switched linear systems

In this section, we consider the following question: "for any fixed p and a given path-complete p-dominant SLS, how can we compute a path-complete automaton and a set of contracting quadratic p-cones that will allow us to certify that the system is path-complete p-dominant?" The section is organized as follows: in Subsection 3.1, we describe an algorithm to compute a set of contracting quadratic p-cones when the automaton and the set of rates are given; in Subsection 3.2, we address the problem of computing the automaton and the set of rates; finally, in Subsection 3.3, we discuss the application and the complexity of the overall algorithmic framework.

3.1 Description of the algorithm

Consider System (1) and let $\mathbf{Aut} = (Q, \Sigma, \delta)$ be a path-complete automaton for the system. Let $\{\gamma_d\}_{d\in\delta}$ be a set of positive rates. Then, according to Definition 2, verifying that (1) is p-dominant with respect to $\mathbf{Aut}$ and $\{\gamma_d\}_{d\in\delta}$ can be addressed by solving the following optimization problem:

$$\max_{\{P_q\}_{q\in Q} \subseteq \mathbb{S}^{n\times n},\, \varepsilon\in\mathbb{R}} \quad \varepsilon \tag{6a}$$
$$\text{s.t.} \quad A_{\mathsf{i}(d)}^\top P_{\mathsf{t}(d)} A_{\mathsf{i}(d)} - \gamma_d^2\, P_{\mathsf{s}(d)} \preceq -\varepsilon I, \;\; \forall\, d \in \delta, \tag{6b}$$
$$-I \preceq P_q \preceq I, \;\; \forall\, q \in Q, \tag{6c}$$
$$P_q \in \mathbb{S}_p^{n\times n}, \;\; \forall\, q \in Q. \tag{6d}$$

The subproblem (6a)–(6c) is a semidefinite optimization problem. Semidefinite programming has become a standard tool in control theory (see, e.g., Boyd et al., 1994) and many different solvers are available to solve these problems in polynomial time (see, e.g., Nesterov and Nemirovskii, 1994, Ben-Tal and Nemirovski, 2001, and Boyd and Vandenberghe, 2004). Unfortunately, the constraints (6d) on the inertia of P_q cannot be expressed as a semidefinite constraint (it is actually nonconvex). However, as we will see below, this set of constraints can in fact be dropped without any impact on the outcome of the decision problem "is System (1) p-dominant with respect to $\mathbf{Aut}$ and $\{\gamma_d\}_{d\in\delta}$?" This statement is formalized in Corollary 13 below. To simplify its presentation, let us make the following assumption on the automaton $\mathbf{Aut}$, without loss of generality (see, e.g., Lind and Marcus, 1995, Proposition 2.2.10):

Assumption 11. *We assume that $\mathbf{Aut} = (Q, \Sigma, \delta)$ is* essential, *meaning that every state $q \in Q$ has an incoming and an outgoing transition: i.e., there are $q_-, q_+ \in Q$ and $i_-, i_+ \in \Sigma$ such that $(q_-, i_-, q) \in \delta$ and $(q, i_+, q_+) \in \delta$.*

The following theorem is the second main result of this paper. It states that either there is no solution of (6a)–(6c) with $\varepsilon > 0$ and with $\{P_q\}_{q\in Q}$ having uniform inertia, or all solutions of (6a)–(6c) with $\varepsilon > 0$ have matrices $\{P_q\}_{q\in Q}$ with the same inertia.

Theorem 12. *Let Assumption 11 hold. Assume there is $k \in \{0, \dots, n\}$ and a feasible solution $(\{P_q\}_{q\in Q}, \varepsilon)$ of (6b)–(6c) with $\varepsilon > 0$ and $\{P_q\}_{q\in Q} \subseteq \mathbb{S}^{n\times n}_k$. Then, it holds that* every *feasible solution $(\{P'_q\}_{q\in Q}, \varepsilon')$ of (6b)–(6c) with $\varepsilon' > 0$ satisfies that $\{P'_q\}_{q\in Q} \subseteq \mathbb{S}^{n\times n}_k$.*

It follows that to verify that (1) is p-dominant with respect to **Aut** and the rates $\{\gamma_d\}_{d\in\delta}$, it suffices to solve the semidefinite optimization problem (6a)–(6c).

Corollary 13. *Under Assumption 11, any optimal solution $(\{P^\star_q\}_{q\in Q}, \varepsilon^\star)$ of (6a)–(6c) satisfies $\varepsilon^\star > 0$ and $\{P^\star_q\}_{q\in Q} \subseteq \mathbb{S}^{n\times n}_p$* if and only if *System (1) is p-dominant with respect to* **Aut** *and $\{\gamma_d\}_{d\in\delta}$.*

Corollary 13 shows that, if the automaton and the rates are given, then the verification of p-dominance for a given SLS can be reduced to a semidefinite optimization problem, and thus can be solved efficiently (see Subsection 3.3 for a discussion of the complexity). Moreover, by Theorem 7, we know that if the system admits a dominated p-splitting, then there is a path-complete automaton and a set of rates for which the system is p-dominant. However, nothing is said about the difficulty of computing this automaton and the associated rates. This question is discussed in the next subsection.

3.2 Constraints on the automaton and the set of rates

There is in general no systematic way to find an automaton and a set of rates that will satisfy the dissipation inequalities (2); see also Subsection 3.3 below. In some cases, the structure of the problem can help us to guess what the automaton and the set of rates will be (some practical examples are given below). When a complete determination of these parameters is not feasible from the structure of the problem, it is nevertheless possible to reduce to "search space" by using the fact that the automaton and the set of rates must satisfy some constraints, as explained below.

To do this, let us consider an automaton $\textbf{Aut} = (Q, \Sigma, \delta)$ and a set of positive rates $\{\gamma_d\}_{d\in\delta}$. Assume that System (1) is p-dominant with respect to **Aut** and $\{\gamma_d\}_{d\in\delta}$, and let $\{P_q\}_{q\in Q}$ be any set of matrices in $\mathbb{S}^{n\times n}_p$ such that (2) holds. We will derive constraints, depending on the matrices of (1), that must be satisfied by $\{\gamma_d\}_{d\in\delta}$ and $\{P_q\}_{q\in Q}$.[4]

[4] These constraints follow in fact from the observation that any cycle in **Aut** defines a p-dominant *LTI* system (see also Subsubsection 2.1.1 and the proofs in the appendix).

Proposition 14. *With* **Aut***, $\{\gamma_d\}_{d\in\delta}$, $\{P_q\}_{q\in Q}$ as above, let $(d_t)_{t=0}^{N-1}$ be a cycle in* **Aut***, $\Phi = A_{\mathsf{i}(d_{N-1})} \cdots A_{\mathsf{i}(d_0)}$ and $\eta = \gamma_{d_0} \cdots \gamma_{d_{N-1}}$. Then, it holds that*

- *(i) The matrix Φ has p eigenvalues with modulus $|\lambda_i| > \eta$ and $n-p$ eigenvalues with modulus $|\lambda_i| < \eta$;*
- *(ii) the eigenspace associated to the p eigenvalues of Φ with modulus $|\lambda_i| > \eta$ is contained in $\mathcal{K}(P_{\mathsf{s}(d_0)})$, and the eigenspace associated to the other $n-p$ eigenvalues is contained in $\mathbb{R}^n \setminus \operatorname{int} \mathcal{K}(P_{\mathsf{s}(d_0)})$.*

Item (i) above is particularly useful to reduce the search space for the rates if the automaton is given. Item (ii) is useful to exclude automata that cannot satisfy the dissipation inequalities (2), for any set of rates. The two examples below illustrate the use of Propositions 14 for the selection of the automaton and of the set of rates. For instance, Example 16 shows that the system of Example 4 cannot be 1-dominant with respect to a single quadratic 1-cone, i.e., with respect to the automaton with a single node.

Example 15. In Example 4, we have used the set of rates $\gamma_d = 0.32$ for all $d \in \delta$ to show that the system is 1-dominant with respect to the automaton $\textbf{Aut}_2$ in Figure 1. These values of the rates were somehow the most natural choice regarding the constraints obtained from Proposition 14-(i) when p is fixed to 1:

- The rate associated to the loop $(\mathtt{a}, 1, \mathtt{a})$ must satisfy $\lambda_1(A_1) = 1 > \gamma_{\mathtt{a1a}} > \lambda_2(A_1) = 0.1$. In the example, we have used the geometric mean of the bounds: $\gamma_{\mathtt{a1a}} = \bar\gamma := \sqrt{0.1}$. Similarly, we have used $\gamma_{\mathtt{b2b}} = \bar\gamma$ for the rate associated to $(\mathtt{b}, 2, \mathtt{b})$.
- By looking at the cycle $(\mathtt{a}, 2, \mathtt{b}, 1, \mathtt{a})$, we get that the associated rates must satisfy $|\lambda_1(A_1A_2)| \approx 0.6 > \gamma_{\mathtt{a2b}}\gamma_{\mathtt{b2a}} > |\lambda_2(A_1A_2)| \approx 0.017$. In the example, we have used $\gamma_{\mathtt{a2b}} = \gamma_{\mathtt{b1a}} = \sqrt[4]{0.6 \cdot 0.017}$ (which in this case can be shown to be equal to $\bar\gamma$).

See also Figure 4-(a) for a representation of the eigenvalues of A_1, A_2, their product and $\bar\gamma$. Note that these rates are not the only ones satisfying the above constraints and that 1-dominance of the system with respect to this set of rates was not guaranteed a priori, but it happened to be the case for this example.

When α increases, the eigenvalues of A_1A_2 (and A_2A_1) get closer to each other; see Figure 4-(c). For $\alpha < 3 - 2\sqrt{2} \approx 0.1716$, the system is still 1-dominant with respect to the same automaton as above and with the rates chosen in the same way as above. However, the contraction property (3) gets more "fragile", in the sense that the images of $\mathcal{K}(P_\mathtt{a})$ and $\mathcal{K}(P_\mathtt{b})$ get closer to the boundary of the cones; see Figure 4-(d). When $\alpha \geq 3-2\sqrt{2}$, the system is not path-complete 1-dominant anymore since the matrix A_1A_2 has two complex conjugated eigenvalues (hence with the same modulus). ◁

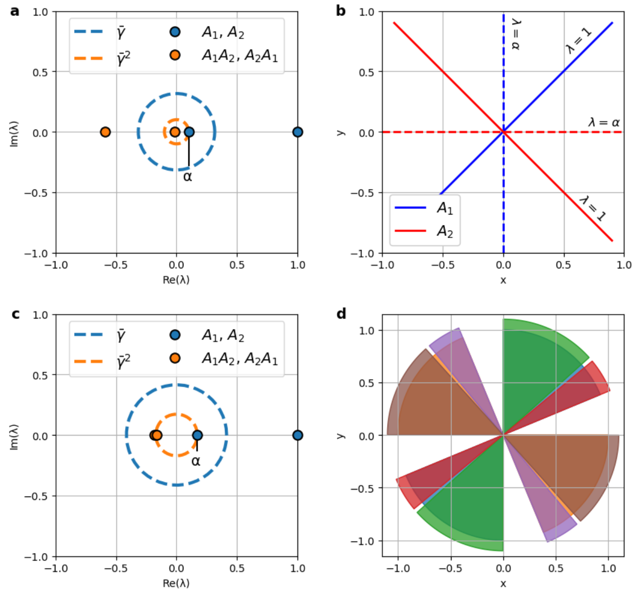

Fig. 4. **a**: Eigenvalues of A_1, A_2, A_1A_2 (see Example 4). **b**: Eigenvectors of A_1 and A_2, associated to $\lambda_1 = 1$ and $\lambda_2 = \alpha$. **c–d**: A_1 and A_2 are as in Example 4 with $\alpha = 0.1715$. **c**: Eigenvalues of A_1, A_2, A_1A_2 and A_2A_1. **d**: Quadratic 1-cones $\mathcal{K}(P_{\mathsf{a}})$ and $\mathcal{K}(P_{\mathsf{b}})$ and their images by A_1 and A_2 (the color code is the same as in Figure 2).

Example 16. From Proposition 14-(ii), it follows that the system of Example 4 cannot be 1-dominant with respect to the automaton $\mathbf{Aut}_1$ in Figure 1 (for any set of rates). Indeed, if it was the case, then the cone $\mathcal{K}(P_{\mathsf{a}})$ would contain the dominant eigenvectors of A_1 and A_2. Because $\mathcal{K}(P_{\mathsf{a}})$ consists of two convex components, this would imply that $\mathcal{K}(P_{\mathsf{a}})$ also contains the eigenvectors associated to $\lambda_2 = \alpha$ of A_1 or A_2 (one can readily check on Figure 4-(b) that any quadratic 1-cone containing the two dominant eigenspaces (solid lines) will also contain one of the dominated eigenspaces (dashed lines)), a contradiction with (ii) in Proposition 14. ◁

3.3 Complexity and comparison with the literature

For a given automaton and a given set of rates, the verification of p-dominance with respect to this automaton and this set of rates can be computed efficiently using Corollary 13. The complexity, using, e.g., interior-point algorithms, is in $\mathcal{O}(|Q|^2|\delta|^{1.5}n^{6.5})$, where $|Q|$ and $|\delta|$ are the number of nodes and the number of transitions in the automaton, and n is the dimension of the system (Ben-Tal and Nemirovski, 2001, Section 6.6.3).

On the other hand, there is no automatic way to find—if it exists—an automaton and a set of rates for which the system is p-dominant. Moreover, the automaton and the rates must be found "all at once", as it is not possible in general to build the automaton and find the associated rates incrementally. Last but not least, there is no upper bound on the size of the automata for which the system is p-dominant, if it is; and thus one may not know when to stop searching for a suitable automaton and conclude that the system is not p-dominant.

These rather deceptive results must be contrasted with the following two observations. The first one is that the problem of p-dominance verification is a difficult problem in itself, as it supersedes the problem of stability of SLSs, which is known to be undecidable (Tsitsiklis and Blondel, 1997). Thus, one may not hope to have a complete, let alone efficient, algorithm for the verification of p-dominance of SLSs, in general.

The second one is that, despite these negative theoretical results, it appears that in many practical situations, a suitable automaton can be easily guessed from the structure of the problem and from Proposition 14-(ii). Similarly, the search space for the rates can be considerably reduced by using the symmetry of the problem (present in many applications) and Proposition 14-(i). As a consequence, finding the automaton and the rates was *not a serious limitation* in the various numerical examples presented in the paper.

The question of numerical verification of the property of having a low-dimensional asymptotic behavior seems to have not received much attention so far. Some contributions in that direction have been made for the specific case of 1-dominance in the works by Forni et al. (2017) and Brundu and Zennaro (2019). The first one presents an algorithm for constructing a *common* convex cone that is contracted by the system. Unfortunately, the restriction to a single common cone adds conservatism to the approach, so that it is not able to capture every SLS with a 1-dimensional asymptotic behavior. The second one describes an algorithm for computing an invariant multicone for SLSs, with invertible matrices, that have a 1-dimensional asymptotic behavior. However, because the computed multicone is not strictly invariant, this approach does not allow to deduce that the system admits a dominated 1-splitting.[5] Closer to our work, the algorithmic verification of p-dominance, with p general, was addressed by Forni and Sepulchre (2019) for continuous-time nonlinear dynamics. The existence of a *common* quadratic p-cone contracted by the system is formulated as the feasibility of a set of LMIs. The concept of path-complete p-dominance introduced here extends this property to discrete-time SLSs and to *families* of quadratic p-cones whose contraction properties

[5] Let us also mention that the approach used in Forni et al. (2017) and Brundu and Zennaro (2019) for the computation of the cone/multicone—which relies on polyhedral set methods, thriving on the fact that the involved sets can be described as the finite union of disjoint convex polyhedral cones—is hardly generalizable to the verification of p-dominance with $p \geq 2$. Indeed, cones that are compatible with p-dimensional attractors are in general not representable as the finite union of convex cones.

$$\begin{bmatrix} 0.1 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0.95 & 0 & 0 & 0.27 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0.255 & 0 \\ 0 & 0 & 0 & 0.21 & 0.63 & 0.49 \\ 0 & 0 & 0 & 0.63 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.595 & 0 \end{bmatrix} \begin{bmatrix} 0.07 & 0.14 & 0.14 & 0 & 0 & 0 \\ 0.665 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.63 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.9 & 0.7 \\ 0.285 & 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0.27 & 0 & 0 & 0.85 & 0 \end{bmatrix}$$

Fig. 5. *Left*: A_1. *Right*: A_2 (see Subsection 4.1).

are dictated by an automaton. Finally, let us mention that a similar approach can be used for the computation of hyperbolicity of discrete-time nonlinear systems (see Berger and Jungers, 2020a).

4 Numerical examples and applications

4.1 1-dominance and population dynamics

Consider the six-dimensional SLS with matrices A_1 and A_2 given in Figure 5. This system, adapted from Schmidbauer et al. (2012, Eq. 4), may appear for instance in the study of *aged-structured populations* with migration between the populations. In this example, $x_1(t)$, $x_2(t)$, $x_3(t)$ (resp. $x_4(t)$, $x_5(t)$, $x_6(t)$), represent the number of individuals in each of the three age classes of some urban (resp. rural) population at time t. Each population evolves according to the Leslie model (see, e.g., Farina and Rinaldi, 2000, and Schmidbauer et al., 2012), and there is migration either from villages to cities (A_1) or from cities to villages (A_2).[6] A central question in the study of population dynamics is whether the asymptotic *composition* of the population depends on the initial value of the population; see, e.g., Farina and Rinaldi (2000), Golubitsky et al. (1975), Tuljapurkar (1982) and Schmidbauer et al. (2012). The population composition is represented by the normalized vector $x(t)/\|x(t)\|_1$. Using dominance analysis, we will show that for any given sequence of matrices A_1 and A_2, $x(t)/\|x(t)\|_1$ is ultimately independent of $x(0)$.

To do this, we consider the automaton $\textsf{\textbf{Aut}}_1$ in Figure 1, together with the set of rates $\gamma_{\mathtt{a1a}} = 0.79$ and $\gamma_{\mathtt{a2a}} = 0.95$ (these rates were selected in the same way as explained in Example 15). Using the algorithm described in Subsection 3.1, we can show that the system is 1-dominant with respect to this automaton and this set of rates. Thus, by Theorem 7, we may conclude that the normalized trajectories of the system converge to the same trajectory. In other words, for any given sequence of matrices A_1 and A_2), the asymptotic composition of the population is independent of its initial value. This is illustrated in Figure 6, where a random sequence of matrices was chosen, and we observe that the different trajectories, starting from different initial conditions, have ultimately the same population composition.

Note that the automaton $\textsf{\textbf{Aut}}_1$ and the above set of rates are not the only ones satisfying the constraints (2) of p-dominance and, even if the system has a 1-dimensional asymptotic behavior, it was not guaranteed a priori that the system is 1-dominant with respect to this automaton and this set of rates. If it had not been the case, then one would have needed to search for more complex automata (like $\textsf{\textbf{Aut}}_2$ or $\textsf{\textbf{Aut}}_3$ for instance). This would have increased the complexity of the problem, but not the conclusion on the asymptotic behavior of the system.

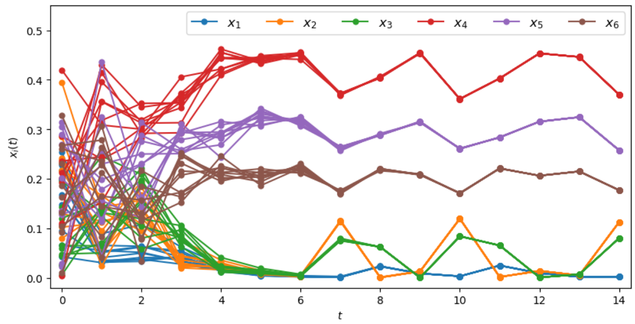

Fig. 6. Normalized trajectories of the system, starting from different initial conditions and for a random sequence of matrices A_1 and A_2. The trajectories are normalized such that $\sum_i x_i(t) = 1$. We observe that all normalized trajectories converge to the same trajectory when $t \to \infty$.

4.2 2-dominant nonlinear system

Consider the discrete-time nonlinear system $x^+ = f(x)$ defined by: (discrete-time Duffing oscillator actuated by a DC motor, adapted from Forni and Sepulchre, 2019)

$$\begin{cases} {x_1}^+ = x_1 + 0.3x_2, \\ {x_2}^+ = 0.3\sin(x_1) - 0.15x_1 + 0.7x_2 + 0.03x_3, \\ {x_3}^+ = -1.5x_1 + 0.925x_3. \end{cases} \tag{7}$$

We will use dominance analysis to show that the asymptotic behavior of any bounded trajectory of (7) is at most 2-dimensional, in the sense that their *ω-limit set* (see, e.g., Khalil, 2002) is contained in a 2-dimensional manifold. This is achieved by considering the *linearized* (aka. *extended*) system:

$$x(t+1) = f(x(t)), \qquad \delta x(t+1) = \partial f_{x(t)} \delta x(t), \tag{8}$$

where ∂f_x is the Jacobian of f at x, which describes the evolution of the sensitivity ($\delta x(t)$) of $x(t)$ to the initial condition $x(0)$. The second equation of (8) is a linear

[6] This is where our model differs from Schmidbauer et al. (2012): instead of having a single matrix that encodes at the same time the migrations from villages to cities and from cities to villages, we have decomposed this matrix in two matrices, A_1 and A_2, to get an SLS.

system whose transition matrix depends on the state of the system. It can thus be abstracted by the SLS with set of matrices $\mathcal{A} = \{\partial f_x : x \in \mathbb{R}^3\}$ (this SLS consists in an *infinite* number of matrices but this will not be a problem for our analysis, as we will see below). We will show that this SLS is path-complete 2-dominant. Then, we will explain the consequence on the asymptotic behavior of bounded trajectories of (7).

To show that the SLS is path-complete 2-dominant, we first note that ∂f_x depends on x only via $\cos(x_1)$, and we partition the set $\mathcal{A}$ into four subsets: $\mathcal{A}_i = \{\partial f_x : x \in \mathbb{R}^3,\ \cos(x_1) \in I_i\}$, $1 \le i \le 4$, where $I_1 = [-1, -\frac{1}{2}]$, $I_2 = [-\frac{1}{2}, 0]$, $I_3 = [0, \frac{1}{2}]$, $I_4 = [\frac{1}{2}, 1]$. Based on these sets of matrices, we consider the automaton depicted in Figure 7-(a) which is path-complete for the SLS defined by $\mathcal{A}$, [7] and we consider the set of rates $\{\gamma_d\}_{d\in\delta}$ given by $\gamma_d = \bar{\gamma} := 0.83$. These rates were selected by trying different values for $\bar{\gamma}$ satisfying the constraints in Proposition 14-(i); see also Figure 7-(b). By using a modification of the algorithm presented in Subsection 3.1, accounting for the fact that the edges are labeled with *sets* of matrices (see Appendix D.1), it can be shown that the SLS defined by $\mathcal{A}$ is 2-dominant with respect to this automaton and this set of rates; and the quadratic 2-cones associated to the symmetric matrices $\{P_q\}_{q\in Q}$ computed with the algorithm are represented in Figure 7-(c). Note also that the automaton is cycle-stable with respect to $\{\gamma_d\}_{d\in\delta}$, since $\bar{\gamma} < 1$. Hence, the SLS defined by $\mathcal{A}$ admits a dominated 2-splitting and is stable on the dominated component of the splitting (see Theorem 10).

Using the above, we may show that the ω-limit set of any bounded trajectory of (7) is at most 2-dimensional. This follows from the following observation:

Proposition 17. *With $\{P_q\}_{q\in Q}$ as above, if y, z are two points in the ω-limit set Ω of some bounded trajectory $x(\cdot)$ of* (7)*, then $z - y$ must belong to at least one of the quadratic* 2*-cones $\mathcal{K}(P_q)$.*

The proof relies on the cone contraction property (3) of p-dominance, the cycle-stability of the automaton and the fact that each $\mathcal{K}(P_q)$ includes a common 2D plane (e.g., the x_1x_2-plane; see Figure 7-(c)), which implies that if $z - y$ does not belong to any $\mathcal{K}(P_q)$, then the pre-image of the line segment joining y to z by f^T is a curve whose length grows exponentially with $T \in \mathbb{N}$, a contradiction with the assumption that $x(\cdot)$ is bounded (thus Ω is compact); see Appendix D.2 for the details. Proposition 17 implies that Ω must lie in a 2-dimensional manifold: indeed, since the x_3-axis is not contained in any $\mathcal{K}(P_q)$ (see Figure 7-(c)), it follows from Proposition 17 that the projection of Ω on the x_1x_2-plane is injective, and thus Ω is at most 2-dimensional.

[7] The transitions are labeled with *sets* of matrices but the principle remains the same: any sequence of matrices in $\mathcal{A}$ can be generated by following a path in the automaton and taking one matrix in the set $\mathcal{A}_i$ associated to each transition of the path.

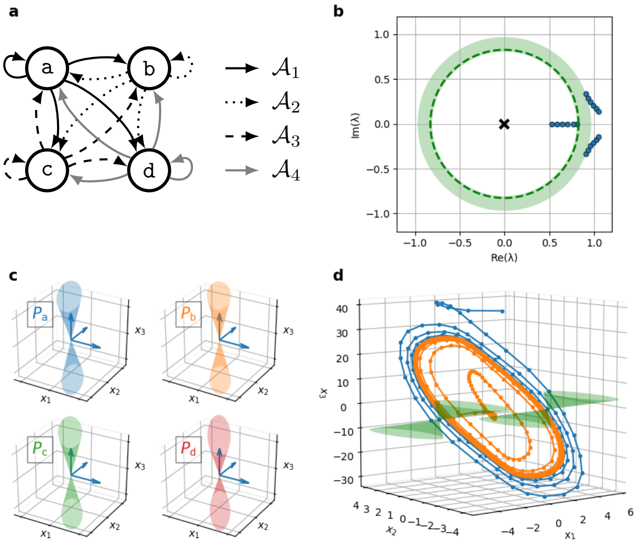

Fig. 7. **a**: Automaton used in Subsection 4.2 (the transitions are labeled according to the legend on the right). **b**: The blue dots represent the eigenvalues of 6 randomly selected matrices in $\mathcal{A}$. According to Proposition 14-(i), the value of $\bar{\gamma}$ must lie in the green strip. **c**: The surfaces represent the boundary of the quadratic 2-cones $\mathcal{K}(P_q)$. Each surface divides the state space into three regions; $\mathcal{K}(P_q)$ is the region of the state space that contains the horizontal plane. **d**: Two trajectories of the system (in blue and orange) and the quadratic 2-cone $\mathcal{K}(P_{\mathsf{c}})$ (in green) centered at 2 different points of the ω-limit set of the trajectory in blue.

To illustrate the above, we have represented in Figure 7-(d) two trajectories of the system, starting from random initial conditions. We verify that the ω-limit set of each trajectory is at most 2-dimensional. We have also represented the cone $\mathcal{K}(P_{\mathsf{c}})$ centered at 2 different points of the ω-limit set of the trajectory in blue. We observe that these cones do not intersect the ω-limit set, as predicted by Proposition 17.

5 Conclusions

In this work, we introduced the concept of path-complete p-dominant SLSs, characterized by the existence of a family of quadratic cones (described by symmetric matrices) whose contraction properties (captured by matrix inequalities) are dictated by an automaton that can generate all switching signals of the system. The goal was to study SLSs with a low-dimensional asymptotic behavior (formalized with the concept of dominated splitting). In particular, we showed that path-complete p-dominant SLSs are exactly the ones that have a p-dimensional asymptotic behavior. Moreover, thriving on the description using symmetric matrices and matrix inequalities, we showed that the property of p-dominance can be formulated as the feasibility of a semidefinite optimization problem, and thus can be verified algorithmically. This allowed us to provide a computational framework for the analysis of SLSs and nonlinear systems with a low-dimensional asymptotic behavior, as demonstrated with several numerical examples.

For future work, we plan to investigate the integration of the concept of p-dominance for other problems in control. We think for instance to the *bisimulation* (aka. *abstraction*) of nonlinear systems (Tabuada, 2009), or the computation of the *estimation entropy* of SLSs (Berger and Jungers, 2020b) and nonlinear systems (Matveev and Pogromsky, 2016). Indeed, an important limitation of the techniques available to address these problems is that they do not scale well with the dimension of the system, *even if the system has a low-dimensional asymptotic behavior* because this information is not used properly. We plan to bridges these gaps by using the algorithmic tools of p-dominance analysis, and show that these techniques can dramatically increase the scalability of these techniques, at least for some classes of systems. We believe that the example of Subsection 4.2 is a proof-of-concept that the theory developed here may have an impact on the numerical and theoretical analysis of nonlinear complex systems, beyond the theory of switched systems.

Acknowledgements

The authors would like to thank Fulvio Forni for insightful discussions on the results presented in the paper and in preliminaries versions of this work.

A Results from linear algebra

Notation. For $P \in \mathbb{S}^{n\times n}$, we let $\nu(P)$ be the number of negative eigenvalues of P, and $\nu_0(P)$ the number of nonpositive eigenvalues of P.

Theorem 18 (Sylvester inertia theorem; see, e.g., Horn and Johnson, 1985, Section 4.5)**.** *Let $Q = A^\top PA$ where $P \in \mathbb{S}^{n\times n}$ and $A \in \mathbb{R}^{n\times n}$. Then, $\nu(Q) \leq \nu(P)$ and $\nu_0(Q) \geq \nu_0(P)$.* [8]

Theorem 19 (Min-max principle; see, e.g., Horn and Johnson, 1985, Section 4.2)**.** *Let $P \in \mathbb{S}^{n\times n}$ and $k \in \{0,\ldots,n\}$. Then, $\nu(P) \geq k$ (resp. $\nu_0(P) \geq k$) if and only if there is a subspace $H \subseteq \mathbb{R}^n$ with dimension k such that $x^\top Px < 0$ (resp. ≤ 0) for all $x \in H \setminus \{0\}$.*

[8] The proof in Horn and Johnson (1985) is presented for A invertible with the conclusion that $\nu(Q) = \nu(P)$ and $\nu_0(Q) = \nu_0(P)$. The case of A singular follows by applying a small perturbation on A and using the continuous dependence of the eigenvalues of symmetric matrices.

Theorem 20 (Main inertia theorem; see, e.g., Lancaster and Tismenetsky, 1985, Section 13.2)**.** *Let $A \in \mathbb{R}^{n\times n}$. There is $P \in \mathbb{S}^{n\times n}$ satisfying $A^\top PA - P \prec 0$ if and only if A has no eigenvalue with modulus $|\lambda_i| = 1$. Moreover, in this case, $P \in \mathbb{S}^{n\times n}_p$ where p is the number of eigenvalues of A with modulus $|\lambda_i| > 1$.*

B Proofs of Section 2

B.1 Proof of Theorem 7. Part 1: (a) ⇒ (b)

Assuming (a), let Aut and $\{\gamma_d\}_{d\in\delta}$ be as in (a), and let $\{P_q\}_{q\in Q}$ be as in Definition 2 for Aut and $\{\gamma_d\}_{d\in\delta}$. Let $\varepsilon > 0$ be small enough so that the right-hand side terms of (2) (in Definition 2) can be replaced by $-\varepsilon I$. Fix $\sigma \in \Sigma^{\mathbb{N}}$. We will build a p-splitting $(\mathcal{E},\mathcal{F})$ which satisfies (5) for some $C \geq 1$ and $\mu \in (0,1)$ independent of σ. To do this, the following notation will be useful: σ being fixed, for $s,t \in \mathbb{N}$, $s < t$, we let $\Phi_{t,s} = A_{\sigma(t-1)}A_{\sigma(t-2)}\cdots A_{\sigma(s)}$. If $s = t$, we let $\Phi_{t,s} = I$. Any trajectory $x(\cdot)$ of (1) with s.s. σ satisfies $x(t) = \Phi_{t,s}x(s)$.

To build the p-splitting $(\mathcal{E},\mathcal{F})$, let $(d_t)_{t=0}^\infty$ be a path in Aut such that $\sigma(t) = \mathsf{i}(d_t)$ for all $t \in \mathbb{N}$. For each $t \in \mathbb{N}$, let $q_t = \mathsf{s}(d_t)$, and for each $q \in Q$, let $V_q(x) = x^\top P_q x$. Remember that (2) implies that

$$V_{q_{t+1}}(A_{\sigma(t)}x) \leq \gamma_{d_t}^2 V_{q_t}(x) - \varepsilon\|x\|^2 \quad \forall\, x \in \mathbb{R}^n. \quad \text{(B.1)}$$

The component $\mathcal{F}$ is defined as follows. Let F_0 be any p-dimensional subspace satisfying $x \in F_0 \Rightarrow V_{q_0}(x) \leq 0$ (see Theorem 19). Then, define the subspaces $\{F_t\}_{t>0}$ as follows: $F_t = \Phi_{t,0}F_0$ for all $t \in \mathbb{N}$. By (B.1), it holds that for every $t \in \mathbb{N}_{>0}$ and $x \in F_0 \setminus \{0\}$, $V_{q_t}(\Phi_{t,0}x) < 0$. This implies that $\operatorname{Ker}\Phi_{t,0} \cap F_0 = \{0\}$, whence F_t has the same dimension as F_0: i.e., $\dim F_t = p$ for all $t \in \mathbb{N}$. The other component $\mathcal{E}$ is defined as follows. For each $s,t \in \mathbb{N}$, $s > t$, let $E'_{s,t} = \{x \in \mathbb{R}^n : V_{q_s}(\Phi_{s,t}x) \geq 0\}$, and for each $t \in \mathbb{N}$, define $E_t = \bigcap_{s>t} E'_{s,t}$. We will show that each E_t contains at least one linear subspace with dimension $n-p$ (the proof that each E_t is actually a linear subspace with dimension $n-p$ will be obtained at the very end of this proof). By Theorem 18, $\Phi_{s,t}^\top P_{q_s}\Phi_{s,t}$ has at least $n-p$ nonnegative eigenvalues, and thus by Theorem 19, $E'_{s,t}$ contains at least one linear subspace with dimension $n-p$. Moreover, (B.1) implies that $E'_{s,t}$ is decreasing with respect to s (and for t fixed): $E'_{s+1,t} \subseteq E'_{s,t}$. Hence, with a standard compactness argument (see, e.g., Berger et al., 2018, Lemma 7), it follows that for each fixed t, the intersection $\bigcap_{s>t} E'_{s,t}$ also contains a subspace with dimension $n-p$.

Now, we show that the pair $(\mathcal{E},\mathcal{F})$ defined above satisfies the relation (5) for some $C \geq 1$ and $\mu \in (0,1)$. Therefore, we will need the following lemma (the proof is presented at the end of this subsection):

Lemma 21. *Let* Aut, $\{\gamma_d\}_{d\in\delta}$ *and* $\{P_q\}_{q\in Q}$ *be as above. There is* $\mu \in (0,1)$ *such that for every* $d \in \delta$, $V_{\mathsf{t}(d)}(A_{\mathsf{i}(d)}x) \leq \gamma_d^2 \cdot \min\left\{\mu V_{\mathsf{s}(d)}(x), \frac{1}{\mu}V_{\mathsf{s}(d)}(x)\right\}$.

Let μ be as in Lemma 21, and let $K \geq 0$ be such that $|V_q(x)| \leq K\|x\|^2$ for all $x \in \mathbb{R}^n$, and $q \in Q$ (by boundedness of $\{P_q\}_{q\in Q}$, such K always exists). Let $t \in \mathbb{N}$ and let $y(\cdot)$ be a trajectory of (1) with s.s. σ and with $y(t) \in F_t$. Then, by Lemma 21 and (B.1), for every $s \in \mathbb{N}$, $s > t$,

$$\begin{aligned}-K\|y(s)\|^2 \leq V_{q_s}(y(s)) &\leq \mu^{t+1-s}\Gamma_{t+1,s}^2 V_{q_{t+1}}(y(t+1))\\ &\leq -\mu^{t+1-s}\Gamma_{t+1,s}^2\varepsilon\|y(t)\|^2,\end{aligned}$$

where $\Gamma_{t+1,s} = \gamma_{d_{t+1}}\cdots\gamma_{d_{s-1}}$. Similarly, we get that for every trajectory $x(\cdot)$ of (1) with s.s. σ and with $x(t) \in E_t$, it holds that for every $s \in \mathbb{N}$, $s > t$,

$$\begin{aligned}\varepsilon\|x(s)\|^2 \leq \gamma_{d_s}^2 V_{q_s}(x(s)) &\leq \mu^{s-t}\Gamma_{t,s}^2\gamma_{d_s}^2 V_{q_t}(x(t))\\ &\leq \mu^{s-t}\Gamma_{t,s}^2\gamma_{d_s}^2 K\|x(t)\|^2. \end{aligned}\tag{B.2}$$

Taking the quotient of $\|x(s)\|$ and $\|y(s)\|$, it follows that (5) holds with μ and with $C = \varepsilon^{-1}K\mu^{-1/2}\max_{d\in\delta}\gamma_d^2$. In particular, μ and C are independent of σ and t. Since t is arbitrary, this holds true for every $t \in \mathbb{N}$.

Finally, we use (5) to show that each E_t is a linear subspace with dimension $n-p$. Therefore, fix $t \in \mathbb{N}$ and assume that $\dim(\operatorname{span} E_t) > n-p$. Thus, $(\operatorname{span} E_t) \cap F_t \neq \{0\}$, so there is $\bar{y} \in F_t \setminus \{0\}$ and $\bar{x}_1, \bar{x}_2 \in E_t$ such that $\bar{y} = \bar{x}_1 + \bar{x}_2$. Letting $y(\cdot)$ [resp. $x_1(\cdot), x_2(\cdot)$] be the trajectory of (1) with s.s. σ and with $y(t) = \bar{y}$ [resp. $x_1(t) = \bar{x}_1$ and $x_2(t) = \bar{x}_2$], we have that for every $s \in \mathbb{N}$, $s \geq t$, $\|y(s)\| \leq 2\max\{\|x_1(s)\|, \|x_2(s)\|\}$; a contradiction with (5). Hence, E_t is a linear subspace with dimension $n-p$. This concludes the proof that (a) $\Rightarrow$ (b). ☐

PROOF of Lemma 21. Because Q is finite there is $\alpha > 0$ such that for every $q \in Q$, $-\varepsilon I \preceq \alpha P_q \preceq \varepsilon I$. Hence, the RHS of (2) can be replaced by αP_{q_1} or $-\alpha P_{q_1}$. This concludes the proof, since by the finiteness of δ, there is $\mu \in (0,1)$ such that for every $d \in \delta$, $\mu\gamma_d^2 \leq \gamma_d^2 - \alpha < \gamma_d^2 + \alpha \leq \mu^{-1}\gamma_d^2$. ☐

B.2 Proof of Theorem 7. Part 2: (b) ⇒ (a)

Assuming (b), let $C \geq 1$ and $\mu \in (0,1)$ be as in (b). The proof that (b) $\Rightarrow$ (a) relies on the following technical lemma (Berger and Jungers, 2019, Lemma 6):

Lemma 22. *There is* $T_* \in \mathbb{N}$ *and* $c > 0$ *such that for every s.s.* $\sigma \in \Sigma^{\mathbb{N}}$ *and every* p*-splitting* $(\mathcal{E}, \mathcal{F})$ *satisfying the assertion of (b) with* σ*, it holds that* $\|A_{\sigma(t)}x\| \geq c\|x\|$ *for every* $t \in \mathbb{N}_{\geq T_*}$ *and* $x \in F_t$.

In the following, it will be convenient to describe the decompositions of $\mathbb{R}^n$ induced by a p-splitting $(\mathcal{E}, \mathcal{F})$ with *projection matrices.* More precisely, given a decomposition $E_t \oplus F_t$ of $\mathbb{R}^n$, we define the matrix $R_t \in \mathbb{R}^{n\times n}$ as the projection on F_t parallel to E_t. Note that R_t determines E_t and F_t completely since $\operatorname{Im} R_t = F_t$ and $\operatorname{Ker} R_t = E_t$; in particular, it holds that $\operatorname{rank} R_t = p$. The following proposition, whose proof can be found in Berger and Jungers (2019, Proposition 7) (and is a straightforward consequence of Lemma 22), states that the set of matrices R_t is bounded for all $t \geq T_*$ and all s.s.:

Proposition 23. *Let* T_* *be as in Lemma 22. There is* $M \geq 0$ *such that for every s.s.* $\sigma \in \Sigma^{\mathbb{N}}$ *and every* p*-splitting* $(\mathcal{E}, \mathcal{F})$ *satisfying the assertion of (b) with* σ*, it holds that* $\|R_t\| \leq M$ *for every* $t \in \mathbb{N}_{\geq T_*}$*, where* R_t *is the projection matrix associated to* (E_t, F_t).

Using the above definitions and results, we will build an automaton, a set of rates and a set of symmetric matrices satisfying (a) in Theorem 7.

Therefore, let $T \in \mathbb{N}_{>0}$ be such that $C\mu^T \leq \frac{1}{4}$ and fix $\alpha \in (0, \frac{3}{10})$. Let $\mathcal{R}_M$ be the set of all projection matrices $R \in \mathbb{R}^{n\times n}$ of rank p and with $\|R\| \leq M$, where M is as in Proposition 23. Since this set is relatively compact, there is a *finite* subset $\{S_1, \ldots, S_m\}$ of $\mathcal{R}_M$ that is an "α-cover" of $\mathcal{R}_M$ (meaning that for any $R \in \mathcal{R}_M$, there is $q \in \{1, \ldots, m\}$ such that $\|R - S_q\| \leq \alpha$).

Now, using this set $\{S_1, \ldots, S_m\}$, we build an automaton $\mathsf{Aut}^* = (Q^*, \Sigma^T, \delta^*)$ and a set of symmetric matrices as follows. The alphabet of Aut^* is Σ^T (the set of words of length T over Σ, i.e., $\Sigma^T = \{(i_1, \ldots, i_T) : i_k \in \Sigma\}$). The set of states of Aut is defined by $Q^* = \{1, \ldots, m\}$. Based on this set, for each $q \in Q^*$, we let

$$P_q = -S_q^\top S_q + (I - S_q)^\top (I - S_q) = I - S_q - S_q^\top. \tag{B.3}$$

By construction, P_q is symmetric. Moreover, P_q is negative definite on $\operatorname{Im} P_q$ and positive definite on $\operatorname{Ker} P_q$. Hence, $P_q \in \mathbb{S}_p^{n\times n}$ (Theorem 19). Finally, we define the set $\delta^* \subseteq Q^* \times \Sigma^T \times Q^*$ of transitions in Aut^* as follows: for every $w = (i_1, \ldots, i_T) \in \Sigma^T$ and $q_1, q_2 \in Q^*$, we let $(q_1, w, q_2) \in \delta^*$ if and only if there is $\kappa > 0$ such that $\Phi_w^\top P_{q_2}\Phi_w \prec \kappa^2 P_{q_1}$, where $\Phi_w = A_{i_T}\cdots A_{i_1}$.

We show that every s.s. $\sigma \in \Sigma^{\mathbb{N}}$ can be read as the juxtaposition of words obtained from a path in Aut^*. Therefore, fix $\sigma \in \Sigma^{\mathbb{N}}$ and decompose σ into blocks of length T: that is, $\sigma = w_0w_1w_2\ldots$, where $w_t = \sigma|_{[tT,tT+T)} \in \Sigma^T$. Let $w_{-1} \in \Sigma^{T_*}$, where T_* is as in Lemma 22, and define $\sigma' = w_{-1}\sigma \in \Sigma^{\mathbb{N}}$. Let $(\mathcal{E}, \mathcal{F})$ be a p-splitting satisfying the assertion of (b) with σ', and let $\{R_t\}_{t=0}^\infty$ be the associated sequence of projection matrices. Finally, for each $t \in \mathbb{N}$, let $q_t = \min\{q \in Q^* : \|R_{tT+T_*} - S_q\| \leq \alpha\}$, which always exists since $\|R_{tT+T_*}\| \leq M$ (Proposition 23).

We claim that $(q_t, w_t, q_{t+1}) \in \delta^*$ for every $t \in \mathbb{N}$, which would prove the assertion at the beginning of the above paragraph. To prove this claim, we fix $t \in \mathbb{N}$, and we will show that there is $\kappa > 0$ such that

$$\Phi_{w_t}^\top P_{q_{t+1}} \Phi_{w_t} - \kappa^2 P_{q_t} \prec 0, \tag{B.4}$$

where $\Phi_{w_t} = A_{\sigma(tT+T-1)} \cdots A_{\sigma(tT)}$. Indeed, let κ be any positive number satisfying

$$2 \sup_{\substack{x\in E_{tT+T_*}\\ \|x\|=1}} \|\Phi_{w_t} x\| \le \kappa \le \frac{1}{2} \inf_{\substack{x\in F_{tT+T_*}\\ \|x\|=1}} \|\Phi_{w_t} x\|, \tag{B.5}$$

The existence of κ is ensured by (5) and $C\mu^T \le \frac{1}{4}$. Also, Lemma 22 ensures that the right-hand side term of (B.5) is positive, so that κ can always be chosen to be positive. To show that (B.4) holds, we let $x \in \mathbb{R}^n \setminus \{0\}$ and $y = \Phi_{w_t} x$, and we will show that $y^\top P_{q_{t+1}} y < \kappa^2 x^\top P_{q_t} x$. Therefore, let $x_1 \in F_{tT+T_*}$ and $x_2 \in E_{tT+T_*}$ such that $x = x_1 + x_2$, and let $y_1 = \Phi_{w_t} x_1$ and $y_2 = \Phi_{w_t} x_2$. Then, since $\|R_{tT+T_*} - S_{q_t}\| \le \alpha$ and $\|R_{(t+1)T+T_*} - S_{q_{t+1}}\| \le \alpha$, we get from (B.3) the following relations (we use capital letters, X_1, X_2, Y_1 and Y_2, to denote the norm of the related vectors; e.g., $X_1 = \|x_1\|$):

$$\begin{aligned} x^\top P_{q_t} x &\ge -X_1^2 + X_2^2 - 2\alpha(X_1^2 + X_2^2), \\ y^\top P_{q_{t+1}} y &\le -Y_1^2 + Y_2^2 + 2\alpha(Y_1^2 + Y_2^2). \end{aligned}$$

We also have the relations $Y_1 \ge 2\kappa X_1$ and $Y_2 \le \frac{1}{2}\kappa X_2$ from (B.5). Hence,

$$\begin{aligned} &\kappa^{-2} y^\top P_{q_{t+1}} y - x^\top P_{q_t} x \\ &\quad \le (-1+2\alpha)(\kappa^{-1}Y_1)^2 + (1+2\alpha)(\kappa^{-1}Y_2)^2 \\ &\qquad + (1+2\alpha)X_1^2 + (-1+2\alpha)X_2^2 \\ &\quad \le 4(-1+2\alpha)X_1^2 + (1+2\alpha)X_1^2 \\ &\qquad + \tfrac{1}{4}(1+2\alpha)X_2^2 + (-1+2\alpha)X_2^2 \\ &\quad = (-3+10\alpha)X_1^2 + \tfrac{1}{4}(-3+10\alpha)X_2^2 < 0. \end{aligned}$$

The latter follows from $\alpha < \frac{3}{10}$. This proves (B.4), and thus it follows that $(q_t, w_t, q_{t+1}) \in \delta^*$, proving the claim at the beginning of this paragraph.

Finally, to conclude the proof of the theorem, it remains to show that, from **Aut*** defined in Step 1, we can build a path-complete automaton **Aut** $= (Q, \Sigma, \delta)$ such that (1) is p-dominant with respect to **Aut** and some rates $\{\gamma_d\}_{d\in\delta} \subseteq \mathbb{R}_{>0}$. This is done by splitting each transition $(q_1, w, q_2) \in \delta^*$ of **Aut*** into T sub-transitions (one per symbol of $w \in \Sigma^T$).

More precisely, for each transition $d = (q_1, w, q_2) \in \delta^*$, we add to $Q^* = \{1, \ldots, m\}$ the states $(d, 1), \ldots, (d, T-1)$. This gives the set of states $Q = Q^* \cup (\delta^* \times \{1, \ldots, T-1\})$. Because Q contains states from Q^* and states induced by the transitions in δ^*, we introduce the following unifying notation: for $d = (q_1, w, q_2) \in \delta^*$ and $k \in \{0, \ldots, T\}$, we let $\bar{q}(d, k) = q_1$ if $k = 0$, $\bar{q}(d, k) = (d, k)$ if $1 \le k \le T-1$, and $\bar{q}(d, k) = q_2$ if $k = T$, and for each $k \in \{0, \ldots, T-1\}$, we let $\bar{w}(d, k) = i_{k+1}$, where $w = (i_1, \ldots, i_T)$. Finally, we define the set $\delta \subseteq Q \times \Sigma \times Q$ of transitions in **Aut** as $\delta = \{(\bar{q}(d, k), \bar{w}(d, k), \bar{q}(d, k+1)) : d \in \delta^*,\ 0 \le k \le T-1\}$. By construction, it is clear that **Aut** $= (Q, \Sigma, \delta)$ is path-complete for (1). It remains to show that that (1) is p-dominant with respect to **Aut** and some set of positive rates $\{\gamma_d\}_{d\in\delta}$. Therefore, we build a set of rates $\{\gamma_d\}_{d\in\delta}$ and a set of matrices $\{\bar{P}_q\}_{q\in Q} \subseteq \mathbb{S}_p^{n\times n}$ such that the dissipation inequalities (2) are satisfied.

To do this, fix a transition $d = (q_1, w, q_2) \in \delta^*$ in **Aut*** and let $\eta_d = \kappa^{1/T}$ where $\kappa > 0$ is such that $\Phi_w^\top P_{q_2} \Phi_w \prec \kappa^2 P_{q_1}$; see (B.4). Let $\bar{P}_{\bar{q}(d,0)} = P_{q_1}$ and $\bar{P}_{\bar{q}(d,T)} = P_{q_2}$. Then, for each $k = T-1, T-2, \ldots, 1$, we define the matrices $P_{\bar{q}(d,k)}$ recursively as follows:

$$\bar{P}_{\bar{q}(d,k)} = \eta_d^{-2} A_{\bar{w}(d,k)}^\top \bar{P}_{\bar{q}(d,k+1)} A_{\bar{w}(d,k)} + \theta I$$

with $\theta > 0$. By construction, we have that for all $k \in \{1, \ldots, T-1\}$,

$$A_{\bar{w}(d,k)}^\top \bar{P}_{\bar{q}(d,k+1)} A_{\bar{w}(d,k)} - \eta_d^2 \bar{P}_{\bar{q}(d,k)} \prec 0. \tag{B.6}$$

Now, observe that

$$A_{\bar{w}(d,0)}^\top \bar{P}_{\bar{q}(d,1)} A_{\bar{w}(d,0)} = \eta_d^{2(1-T)} \Phi_w^\top \bar{P}_{\bar{q}(d,T)} \Phi_w + \Delta$$

where $\Delta \in \mathbb{S}^{n\times n}$ satisfies $\|\Delta\| \to 0$ as $\theta \to 0$. Hence, since $A_w^\top \bar{P}_{q_2} A_w \prec \kappa^2 \bar{P}_{q_1}$, it follows that (B.6) also holds for $k = 0$, provided θ is small enough.

Summarizing, we have shown that the automaton **Aut**, together with the rates $\{\gamma_{d'}\}_{d'\in\delta}$ defined by $\gamma_{d'} = \eta_d$ if $d' = (\bar{q}(d,k), \bar{w}(d,k), \bar{q}(d,k+1)) \in \delta$ and with the matrices $\{\bar{P}_q\}_{q\in Q}$ defined as above, satisfies (2). Hence, to show that (a) in Theorem 7 holds, it remains to show that $\{\bar{P}_q\}_{q\in Q} \subseteq \mathbb{S}_p^{n\times n}$. By using (B.6) and Theorem 18, we get that for every $d \in \delta^*$, $p = \nu(\bar{P}_{\bar{q}(d,T)}) \ge \nu_0(\bar{P}_{\bar{q}(d,T-1)}) \ge \nu(\bar{P}_{\bar{q}(d,T-1)}) \ge \ldots \ge \nu_0(\bar{P}_{\bar{q}(d,1)}) \ge \nu(\bar{P}_{\bar{q}(d,1)}) \ge \nu_0(\bar{P}_{\bar{q}(d,0)}) = p$, whence for all $k \in \{0, \ldots, T\}$, $\nu(\bar{P}_{\bar{q}(d,k)}) = \nu_0(\bar{P}_{\bar{q}(d,k)}) = p$, which concludes the proof that (b) ⇒ (a). □

B.3 Proof of Theorem 10. Part 1: (a) ⇒ (b)

Assuming (a), the first assertion in (b) follows directly from Theorem 7. The second assertion in (b) follows from (B.2) and the fact that since **Aut** is cycle-stable with respect to $\{\gamma_d\}_{d\in\delta}$ there is $M \ge 1$ such that $\Gamma_{s,t} \le M$ for every $s, t \in \mathbb{N}$, $s \le t$. Hence, it suffices to take $\rho = \mu^{1/2}$ and $D = (\varepsilon^{-1} K M)^{1/2}$. □

B.4 Proof of Theorem 10. Part 2: (b) ⇒ (a)

The proof is very similar to the proof that (b) ⇒ (a) in Theorem 7 (Subsection B.2). We just need to make the following modifications: *(i)* We let $T \in \mathbb{N}$ be such that $C\mu^T \leq \frac{1}{4}$ and $D\rho^T < \frac{1}{2}$ (the second constraint will imply that there is $\kappa \in (0,1)$ satisfying (B.5)); *(ii)* We let $(q_1, w, q_2) \in \delta^*$ if and only if $\Phi_w^\top P_{q_2} \Phi_w \prec \kappa^2 P_{q_1}$ for some $\kappa \in (0,1)$. The rest of the proof is exactly the same as in Subsection B.2. Observe that since $\kappa < 1$ we have $\eta_d < 1$ and thus $\gamma_{d'} < 1$. Hence, the automaton **Aut** is cycle-stable with respect to $\{\gamma_{d'}\}_{d' \in \delta}$. □

C Proofs of Section 3

C.1 Proof of Theorem 12

We consider an automaton $\mathbf{Aut} = (Q, \Sigma, \delta)$ satisfying Assumption 11. We say that $q \in Q$ is *recurrent* if there is a path $(d_t)_{t=0}^{N-1}$ with length $N \geq 1$ from q to q, i.e., with $\mathsf{s}(d_0) = \mathsf{t}(d_{N-1}) = q$. Let $(\{P_q\}_{q \in Q}, \varepsilon)$ be a feasible solution of (6b)–(6c) with $\varepsilon > 0$.

We will first show that for any recurrent state $q \in Q$ the inertia of P_q depends only on the automaton, the rates and the matrices $\{A_i\}_{i \in \Sigma}$. To show this, fix a recurrent state $q \in Q$ and let $(d_t)_{t=0}^{N-1}$ be a path from q to itself. For each $k \in \{0, \ldots, N-1\}$, let $\Phi_k = A_{\mathsf{i}(d_k)} \cdots A_{\mathsf{i}(d_0)}$ and $\eta_k = \gamma_{d_{N-1}} \cdots \gamma_{d_k}$. Then, from (6b) and using that $P_q = P_{\mathsf{s}(q_0)} = P_{\mathsf{t}(q_{N-1})}$, we get that

$$\begin{aligned} \Phi_{N-1}^\top P_q \Phi_{N-1} &\prec \eta_{N-1}^2 \Phi_{N-2}^\top P_{\mathsf{t}(q_{N-2})} \Phi_{N-2} \\ &\prec \eta_{N-2}^2 \Phi_{N-3}^\top P_{\mathsf{t}(q_{N-3})} \Phi_{N-3} \prec \ldots \prec \eta_0^2 P_q. \end{aligned} \tag{C.1}$$

By Theorem 20, it follows that $P_q \in \mathbb{S}_{k_q}^{n \times n}$ where k_q is the number of eigenvalues of Φ_{N-1} with modulus $|\lambda_i| > \eta_0$. Because Φ_{N-1} and η_0 depend only on **Aut**, $\{\gamma_d\}_{d \in \delta}$, and $\{A_i\}_{i \in \Sigma}$, and by the hypothesis of Theorem 12, it follows that for every recurrent state $q \in Q$, $P_q \in \mathbb{S}_k^{n \times n}$.

Now, let $q \in Q$ be any state. By Assumption 11, there is a recurrent state r and a path $(d_t)_{t=0}^{N-1}$ from r to q (since any backward infinite path from q will eventually loop on itself). By the same argument as above, it holds that $\Phi^\top P_q \Phi - \eta^2 P_r \prec 0$, where $\Phi = A_{\mathsf{i}(d_{N-1})} \cdots A_{\mathsf{i}(d_0)}$ and $\eta = \gamma_{d_{N-1}} \cdots \gamma_{d_0}$. Hence, by Theorem 18, it follows that $\nu(P_q) \geq \nu_0(P_r) = k$. By proceeding in a similar way (using a path from q to a recurrent state), we can show that $\nu_0(P_q) \leq k$. Hence, $\nu(P_q) = \nu_0(P_q) = k$, and thus $P_q \in \mathbb{S}_k^{n \times n}$, concluding the proof of the theorem. □

C.2 Proof of Corollary 13

The "only if" direction is clear: if the optimal solution $(\{P_q^\star\}_{q \in Q}, \varepsilon^\star)$ satisfies the assertions of the corollary, then (2) holds with $\{P_q^\star\}_{q \in Q}$ and thus the system is p-dominant with respect to **Aut** and $\{\gamma_d\}_{d \in \delta}$.

The "if" direction is also straightforward: if the system is p-dominant with respect to **Aut** and $\{\gamma_d\}_{d \in \delta}$, then (6a)–(6c) has a feasible solution with $\varepsilon > 0$ and with $\{P_q\}_{q \in Q} \subseteq \mathbb{S}_p^{n \times n}$. It follows that any optimal solution $(\{P_q^\star\}_{q \in Q}, \varepsilon^\star)$ satisfies $\varepsilon^\star > 0$, and thus, by Theorem 12, it holds that $\{P_q^\star\}_{q \in Q} \subseteq \mathbb{S}_p^{n \times n}$. □

C.3 Proof of Proposition 14

Let $P = P_{\mathsf{s}(d_0)}$. By using the same argument as in (C.1), we get that $\Phi^\top P \Phi - \eta^2 P \prec 0$. Hence, by Theorem 20, Φ has p eigenvalues with modulus $> \eta$ and $n - p$ eigenvalues with modulus $< \eta$. This proves (i). Now, in order to prove (ii), let the columns of $H \in \mathbb{R}^{n \times p}$ be a basis of the eigenspace, denoted by F, associated to the eigenvalues of Φ with modulus $> \eta$. Then, it holds that $\Phi H = H \Phi_p$ for some $\Phi_p \in \mathbb{R}^{p \times p}$ with eigenvalues equal to the eigenvalues of Φ with modulus $> \eta$. It follows that $\Phi_p^\top H^\top P H \Phi_p - \eta^2 H^\top P H \prec 0$, and thus by Theorem 20, $H^\top P H$ is negative definite. This implies that any vector $x \in F$ satisfies $x^\top P x \leq 0$, so that $F \subseteq \mathcal{K}(P)$. A similar reasoning shows that any vector x in the eigenspace associated to the $n - p$ eigenvalues of Φ with modulus $< \eta$ satisfies $x^\top P x \geq 0$. This concludes the proof of (ii). □

D Addendum to Section 4

D.1 Automaton labeled with sets of matrices

Let $\mathcal{A}_i \subseteq \mathbb{R}^{n \times n}$ be sets of matrices indexed by $i \in \Sigma := \{1, \ldots, N\}$. Consider an automaton $\mathbf{Aut} = (Q, \Sigma, \delta)$ and a set of positive rates $\{\gamma_d\}_{q \in \delta}$. Assume that there exists a set of matrices $\{P_q\}_{q \in Q} \subseteq \mathbb{S}_p^{n \times n}$ such that for every $d \in \delta$ and $A \in \mathcal{A}_{\mathsf{i}(d)}$,

$$A^\top P_{\mathsf{t}(d)} A - \gamma_d^2 P_{\mathsf{s}(d)} \prec 0. \tag{D.1}$$

These inequalities generalize the ones of p-dominance to SLSs with an infinite number of matrices. From the proof of (a) ⇒ (b) in Theorem 7 (see Appendix B.1), it is clear that this implication also holds for SLSs with an infinite set of matrices satisfying (D.1).

Regarding the algorithmic verification of (D.1), assume that each matrix set $\mathcal{A}_i$ can be represented as a convex set $\mathcal{A}_i = A_i + \operatorname{conv}\{\Delta_{i,1}, \ldots, \Delta_{i,N_i}\}$ where $A_i, \Delta_{i,1}, \ldots, \Delta_{i,N_i} \in \mathbb{R}^{n \times n}$, and consider the following semidefinite optimization problem:

$$
\begin{aligned}
&\max_{\{P_q\}_{q\in Q}\subseteq\mathbb{S}^{n\times n},\,\{E_q\}_{q\in Q}\subseteq\mathbb{S}^{n\times n},\,\varepsilon\in\mathbb{R}} \quad \varepsilon && \text{(D.2a)}\\
&\text{s.t. } A_{\mathsf{i}(d)}^\top P_{\mathsf{t}(d)} A_{\mathsf{i}(d)} + A_{\mathsf{i}(d)}^\top P_{\mathsf{t}(d)} \Delta_{\mathsf{i}(d),j} + \Delta_{\mathsf{i}(d),j}^\top P_{\mathsf{t}(d)} A_{\mathsf{i}(d)} \\
&\qquad + \Delta_{\mathsf{i}(d),j}^\top E_d \Delta_{\mathsf{i}(d),j} - \gamma_d^2 P_{\mathsf{s}(d)} \preceq -\varepsilon I,\\
&\qquad\qquad \forall\, d\in\delta, j\in\{1,\ldots,N_{\mathsf{i}}(d)\}. && \text{(D.2b)}\\
&\quad -I \preceq P_{\mathsf{t}(d)} \preceq E_d \preceq I,\ E_d\succeq 0,\ \ \forall\, d\in\delta. && \text{(D.2c)}
\end{aligned}
$$

By using a similar argument as in the proof of Theorem 12, we may show that if (D.2) admits a solution $(\{P_q\}_{q\in Q},\{E_q\}_{q\in Q},\varepsilon)$ with $\varepsilon>0$ and $\{P_q\}_{q\in Q}\subseteq\mathbb{S}_p^{n\times n}$, then the system satisfies (D.1) with such P_q and ε. The proof (omitted due to space limitation) is based on the convexity of the set of matrices Δ satisfying (D.2b) with P_q and E_q fixed (this is why we need $E_d\succeq 0$).

D.2 Proof of Proposition 17

Assume, for a contradiction, that $z-y\notin\bigcup_{q\in Q}\mathcal{K}(P_q)$. Fix $T\in\mathbb{N}$, and let $\alpha:[0,1]\to\mathbb{R}^n$ be the line segment from y to z and let $\beta:[0,1]\to\mathbb{R}^n$ be the pre-image of α by f^T (which exists by the Inverse Function theorem; see, e.g., Robinson, 1999, Theorem V.2.4). By invariance of Ω, it holds that $\beta(0)=f^{-T}(y)\in\Omega$ and $\beta(1)=f^{-T}(z)\in\Omega$. Fix $r\in[0,1]$. By the contraction property (3), it holds that $\beta'(r)\notin\bigcap_{q\in Q}\mathcal{K}(P_q)$, as otherwise we would have $\alpha'(r)=\partial f^T(\beta(r))\beta'(r)\in\bigcup_{q\in Q}\mathcal{K}(P_q)$, a contradiction with the definition of α. Moreover, since the automaton is cycle-stable, it holds that $\|\beta'(r)\|\geq D\rho^{-T}\|\alpha'(r)\|$ with $D>0$ and $\rho\in(0,1)$ independent of T and r (Theorem 10). Moreover, from the shape of the cones $\mathcal{K}(P_q)$ which contain the x_1x_2-plane in their interior (see Figure 7-(c)), there is $c>0$, independent of T and r, such that $|e_3^\top\beta'(r)|\geq c\|\beta'(r)\|$, where $e_3=[0,0,1]^\top$. Furthermore, by continuity of $\beta'(r)$ with respect to r, $e_3^\top\beta'(r)$ has the same sign for all r. Thus, by integration of $e_3^\top\beta'$, we find that $|e_3^\top[\beta(1)-\beta(0)]|$ increases exponentially with T. Since T is arbitrarily, this is a contradiction with the boundedness of $x(\cdot)$. □

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