Convex and Conic Optimization

Duality Theorems

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1 INTRODUCTION AND DEFINITIONS

Let n be a fixed positive integer.

Definition 1.1. A set $K \subseteq \mathbb{R}^n$ is a *cone* if for every $x \in K$ and $\alpha \in \mathbb{R}_{\geq 0}$, $\alpha x \in K$.

Definition 1.2. Let $C \subseteq \mathbb{R}^n$. The *dual cone* of C, denoted by C^*, is the set defined by $C^* = \{y \in \mathbb{R}^n : y^\top x \geq 0 \;\forall\, x \in C\}$.

2 DUALITY IN CONIC OPTIMIZATION

Fix $m_1 \in \mathbb{N}$ and $m_2 \in \mathbb{N}$. Fix $c \in \mathbb{R}^n$, $A_1 \in \mathbb{R}^{m_1 \times n}$, $A_2 \in \mathbb{R}^{m_2 \times n}$, $b_1 \in \mathbb{R}^{m_1}$ and $b_2 \in \mathbb{R}^{m_2}$. Let $K \subseteq \mathbb{R}^n$ be a fixed closed convex cone.

Consider the following optimization problem:

$$\begin{array}{lll} \mathbb{P}: & \inf_x & c^\top x \\ & \text{s.t.} & A_1 x = b_1, \\ & & A_2 x \geq b_2, \\ & & x \in K, \end{array} \tag{1}$$

with variable $x \in \mathbb{R}^n$. The dual of (1) is the problem:

$$\begin{array}{lll} \mathbb{D}: & \sup_{y_1, y_2} & b_1^\top y_1 + b_2^\top y_2 \\ & \text{s.t.} & c - A_1^\top y_1 - A_2^\top y_2 \in K^*, \\ & & y_2 \geq 0, \end{array} \tag{2}$$

with variables $y_1 \in \mathbb{R}^{m_1}$ and $y_2 \in \mathbb{R}^{m_2}$.

Proposition 2.1 (Weak duality). *For any feasible solutions x and (y_1, y_2) of $\mathbb{P}$ and $\mathbb{D}$ respectively, it holds that $c^\top x \geq b_1^\top y_1 + b_2^\top y_2$.*

Proof. By definition of K^*, it holds that $c^\top x \geq (y_1^\top A_1 + y_2^\top A_2)x$. Now, since $y_2 \geq 0$, it holds that $(y_1^\top A_1 + y_2^\top A_2)x \geq b_1^\top y_1 + b_2^\top y_2$. □

Corollary 2.2. *If $\mathbb{P}$ is unbounded, then $\mathbb{D}$ is infeasible. If $\mathbb{D}$ is unbounded, then $\mathbb{P}$ is infeasible.*

Definition 2.3. $\mathbb{P}$ is said to be *strictly feasible* if there is $x \in \mathbb{R}^n$ such that $A_1 x = b_1$, $A_2 x \geq b_2$ and $x \in \operatorname{int}(K)$.

Theorem 2.4 (Strong duality). *Assume that $\mathbb{P}$ is strictly feasible and bounded. Then, $\mathbb{D}$ has an optimal solution and* $\inf \mathbb{P} = \max \mathbb{D}$.

Proof. Consider the sets $K_1 = K \times \mathbb{R}_{\geq 0}$ and $K_2 = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} : A_1 x = t b_1,\ A_2 x \geq t b_2\}$. It holds that K_1 and K_2 are closed convex cones. Let $p^* = \inf \mathbb{P}$ and define $d = [c^\top, -p^*]^\top$. It holds that for all $(x,t) \in K_1 \cap K_2$, $d^\top [x^\top, t]^\top \geq 0$. Hence, $d \in (K_1 \cap K_2)^*$. Moreover, since $\operatorname{int}(K_1) \cap K_2 \neq \emptyset$, it holds (Propositions A.3 and A.5) that $(K_1 \cap K_2)^* = K_1^* + K_2^*$. It holds that $K_1^* = K^* \times \mathbb{R}_{\geq 0}$ and

$$K_2^* = \{[y_1^\top A_1, -y_1^\top b_1]^\top + [y_2^\top A_2, -y_2^\top b_2]^\top : y_1 \in \mathbb{R}^{m_1},\ y_2 \in (\mathbb{R}_{\geq 0})^{m_2}\}$$

(Corollary A.4). Hence, there is $y_1 \in \mathbb{R}^{m_1}$ and $y_2 \in (\mathbb{R}_{\geq 0})^{m_2}$ such that $d - [y_1^\top A_1, -y_1^\top b_1]^\top + [y_2^\top A_2, -y_2^\top b_2]^\top \in K_1^*$. The latter is equivalent to $c - A_1^\top y_1 - A_2^\top y_2 \in K^*$ and $p^* \leq b_1^\top y_1 + b_2^\top y_2$, concluding the proof. □

Example 2.5 ($\mathbb{P}$ is strictly feasible and bounded, but has no optimal solution). Let $\mathbb{P}$ have variables $x_1, x_2, x_3 \in \mathbb{R}$, constraints $x_1 x_2 \geq x_3^2$, $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 = 1$, and objective to minimize x_1. Then $\inf \mathbb{P} = 0$, but the infimum cannot be reached because for any point (x_1, x_2, x_3) with $x_1 = 0$, it holds that $x_1 x_2 = 0 \ngeq 1$. The dual of $\mathbb{P}$ is the problem $\mathbb{D}$ with variable $y \in \mathbb{R}$, constraint $1 \cdot 0 \geq (-y)^2$, and objective to maximize y. We see that $\mathbb{D}$ has an optimal solution.

A SOME RESULTS FROM CONVEX ANALYSIS

Proposition A.1. *For any $C \subseteq \mathbb{R}^n$, C^* is closed and convex.*

Proof. Straightforward. □

Proposition A.2. *Let $K \subseteq \mathbb{R}^n$ be a closed convex cone. It holds that $(K^*)^* = K$.*

Proof. It is clear that $K \subseteq (K^*)^*$. Now, we show by contradiction that $(K^*)^* \subseteq K$. Therefore, assume there is $x' \in (K^*)^* \setminus K$. Then, since K is closed and convex, there is $y' \in \mathbb{R}^n$ such that $y'^\top x \geq 0$ for all $x \in K$ and $y'^\top x' < 0$. In other words, $y' \in K^*$ and $y'^\top x' < 0$, a contradiction with $x' \in (K^*)^*$. □

Proposition A.3. *Let $K_1, \ldots, K_m \subseteq \mathbb{R}^n$ be closed convex cones. Assume that $K_1^* + \ldots + K_m^*$ is closed. Then, $(K_1 \cap \ldots \cap K_m)^* = K_1^* + \ldots + K_m^*$.*

Proof. It is clear that $K_1^* + \ldots + K_m^* \subseteq (K_1 \cap \ldots \cap K_m)^*$. Now, we show that $(K_1 \cap \ldots \cap K_m)^* \subseteq K_1^* + \ldots + K_m^*$. Therefore, we show that $(K_1^* + \ldots + K_m^*)^* \subseteq K_1 \cap \ldots \cap K_m$, and we use the fact that $((K_1^* + \ldots + K_m^*)^*)^* = K_1^* + \ldots + K_m^*$ (Proposition A.2) and that, for any $A, B \subseteq \mathbb{R}^n$, $A \subseteq B$ implies $B^* \subseteq A^*$. Let $x \in (K_1^* + \ldots + K_m^*)^*$. Fix $i \in \{1, \ldots, m\}$. It holds that for all $y \in K_i^*$, $y^\top x \geq 0$; hence $x \in (K_i^*)^* = K_i$ (Proposition A.2). Since i was arbitrary, this concludes the proof. □

Corollary A.4. *Let $A \in \mathbb{R}^{m \times n}$ and $K = \{x \in \mathbb{R}^n : Ax \geq 0\}$. It holds that $K^* = \{A^\top y : y \in (\mathbb{R}_{\geq 0})^m\}$.*

Proof. Follows from the fact $K^* = \{A^\top y : y \in (\mathbb{R}_{\geq 0})^m\}$ is closed, as that the conic hull of a finite set of vectors (proof omitted). □

Proposition A.5. *Let $K_1 \subseteq \mathbb{R}^n$ and $K_2 \subseteq \mathbb{R}^n$ be closed convex cones. Assume that $\operatorname{int}(K_1) \cap K_2 \neq \emptyset$. Then, $K_1^* + K_2^*$ is closed.*

Proof. Let $(y_{1,k})_{k \in \mathbb{N}} \subseteq K_1^*$ and $(y_{2,k})_{k \in \mathbb{N}} \subseteq K_2^*$, and let $(y_k)_{k \in \mathbb{N}}$ be defined by $y_k = y_{1,k} + y_{2,k}$. Assume that $y_k \to y_*$. We show that $y_* \in K_1^* + K_2^*$. For that, we show that $(y_{1,k})_{k \in \mathbb{N}}$ is bounded. Indeed, let $x \in \operatorname{int}(K_1) \cap K_2$. There is $D \in \mathbb{R}$ such that for all $k \in \mathbb{N}$, $y_k^\top x = y_{1,k}^\top x + y_{2,k}^\top x \leq D$. Thus, for all $k \in \mathbb{N}$, $y_{k,1}^\top x \leq D$. Since, for all $y' \in K_1^*$, with $\|y'\| = 1$, $y'^\top x > 0$, it follows that $(y_{1,k})_{k \in \mathbb{N}}$ is bounded. Now, since $(y_k)_{k \in \mathbb{N}}$ and $(y_{1,k})_{k \in \mathbb{N}}$ are bounded, $(y_{2,k})_{k \in \mathbb{N}}$ is too. Thus, $(y_{1,k})_{k \in \mathbb{N}}$ and $(y_{2,k})_{k \in \mathbb{N}}$ converge in K_1^* and K_2^* respectively. It follows that $y_* \in K_1^* + K_2^*$. □