CSCI 5654, Spring 2023: Assignment 2

Assigned date: We 2/15/2023
Due date: We 3/1/2023

Instructions: Please upload your HW as a PDF file.

Consider the following Linear Program:

$$\begin{array}{rl} \max & 0x_1 + 0x_2 \\ \text{s.t.} & x_1 - x_2 \le -1 \\ & x_2 \le 3 \\ & -x_1 - x_2 \le -6 \\ & x_1, x_2 \ge 0\ . \end{array} \tag{1}$$

We are interested in finding a feasible solution of (1), if one exists.

Problem 1 (5 points): Primal-Based Approach

First, we consider a "primal-based" approach to compute a feasible solution of (1). Therefore, build the *auxiliary problem*[1] (reminder: the auxiliary problem has an extra variable "x_0" that makes the constraints feasible and the objective is to minimize x_0). Solve the auxiliary problem using Simplex.[2] Deduce a feasible solution of (1), or conclude that (1) is infeasible.

Problem 2 (5 points): Dual-Based Approach

Next, we consider a "dual-based" approach to compute a feasible solution of (1). Therefore, provide the dual problem of (1). Solve the dual problem using Simplex (use *Bland's rule* to tackle degeneracy).[3] Deduce a feasible solution of (1), or conclude that (1) is infeasible.

Problem 3 (5 points): Unboundedness and Parametric Solution

For a generic Linear Program:

$$\begin{array}{rl} \max & c^\top x \\ \text{s.t.} & w = b - Ax \ge 0 \\ & x, w \ge 0\ , \end{array} \tag{2}$$

consider a *feasible* dictionary in which $x_1, \ldots, x_k$ and $w_1, \ldots, w_{n-k}$ are nonbasic (and $x_{k+1}, \ldots, x_n$ and $w_{n-k+1}, \ldots, w_m$ are basic):

$$\begin{array}{llllll} \zeta & = \bar{c}_0 & +\bar{c}_1 x_1 & \ldots +\bar{c}_k x_k & +\bar{c}_{k+1} w_1 & \ldots +\bar{c}_n w_{n-k} \\ \hline x_{k+1} & = \bar{b}_1 & -\bar{a}_{1,1} x_1 & \ldots -\bar{a}_{1,k} x_k & -\bar{a}_{1,k+1} w_1 & \ldots -\bar{a}_{1,n} w_{n-k} \\ \vdots & & & & & \\ x_n & = \bar{b}_{n-k} & -\bar{a}_{n-k,1} x_1 & \ldots -\bar{a}_{n-k,k} x_k & -\bar{a}_{n-k,k+1} w_1 & \ldots -\bar{a}_{n-k,n} w_{n-k} \\ w_{n-k+1} & = \bar{b}_{n-k+1} & -\bar{a}_{n-k+1,1} x_1 & \ldots -\bar{a}_{n-k+1,k} x_k & -\bar{a}_{n-k+1,k+1} w_1 & \ldots -\bar{a}_{n-k+1,n} w_{n-k} \\ \vdots & & & & & \\ w_m & = \bar{b}_m & -\bar{a}_{m,1} x_1 & \ldots -\bar{a}_{m,k} x_k & -\bar{a}_{m,k+1} w_1 & \ldots -\bar{a}_{m,n} w_{n-k} \end{array}$$

(a) Let $j \in \{1, \ldots, n\}$ and assume that $\bar{c}_j > 0$. Give a sufficient condition on $\bar{a}_{1,j}, \ldots, \bar{a}_{m,j}$ for (2) to be unbounded.

(b) With the assumption above (for simplicity, also assume that $j \le k$), give a parametric solution $x(t)$, parameterized by $t \ge 0$, that is feasible for all $t \ge 0$ and such that $c^\top x(t) \to +\infty$ when $t \to +\infty$.[4]

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[1] See Vanderbei's book, page 17.

[2] Explain what are the basic and nonbasic variables in the initial *feasible* dictionary. Write down the dictionaries appearing during the execution of the Simplex before the final one. Give the expression of the objective function as a function of the nonbasic variables in the final dictionary. No more details needed.

[3] Write down the dictionaries appearing during the execution of the Simplex until the final one (included). Explain which variable is entering and which one is leaving. Explain why this dictionary is final. No more details needed.

[4] Hint: use x_j as parameter and express the basic variables $x_{k+1}, \ldots, x_n$ in function of x_j.

(c) Deduce[5] that if (2) is unbounded, then there is a feasible solution x^* that contains at most $m+1$ nonzero values, and such that $c^\top x^* = 1$.

Comment: By applying the above on the dual of a feasibility problem $Ax \le b$, we can show that if $Ax \le b$ is infeasible, then there is a subset of $m+1$ linear inequalities in $Ax \le b$ that are infeasible. (Do not show this.)

Problem 4 (5 points): Dual of Investment Problem

In the previous homework assignment, you solved a portfolio optimization problem. The problem involved N (15) variables x_j corresponding to the amount of stock j to have in portfolio. The price of each stock j was p_j and the return was r_j. The goal was to maximize the total return. There were also constraints to satisfy: namely, for K different subset S_k of stocks (e.g., gathered by risk category, market sector, or ecological impact), the total price of the stocks in S_k had to be between bounds ℓ_k and u_k. In summary, the problem was

$$\begin{array}{rl} \max & \sum_{j=1}^{N} r_j x_j \\ \text{s.t.} & \sum_{j \in S_k} p_j x_j \in [\ell_k, u_k] \quad \forall\, k \in \{1, \ldots, K\} \\ & x_1, \ldots, x_N \ge 0. \end{array}$$

(1) Write the dual of the above problem.[6]

(2) Solve the primal and dual problem, with the data given in the previous homework assignment. Use any software you want. Report the optimal solutions (x^*, w^*) and (z^*, y^*) by completing the `NaN` values in Figure 1. Verify that complementary slackness holds.

[5] You may assume without proof that if (2) is unbounded, then a dictionary satisfying the conditions in (a) will always be reached.

[6] The notation $1_{j \in S_k} \doteq \begin{cases} 1 & \text{if } j \in S_k \\ 0 & \text{if } j \notin S_k \end{cases}$ may be useful.

```
x_{1} = NaN
x_{2} = NaN
x_{3} = NaN
x_{4} = NaN
x_{5} = NaN
x_{6} = NaN
x_{7} = NaN
x_{8} = NaN
x_{9} = NaN
x_{10} = NaN
x_{11} = NaN
x_{12} = NaN
x_{13} = NaN
x_{14} = NaN
x_{15} = NaN
w_{totinvest} = NaN
w_{risk,A,ℓ} = NaN
w_{risk,A,u} = NaN
w_{risk,B,ℓ} = NaN
w_{risk,B,u} = NaN
w_{risk,C,ℓ} = NaN
w_{risk,C,u} = NaN
w_{risk,D,ℓ} = NaN
w_{risk,D,u} = NaN
w_{market,       Tech,ℓ} = NaN
w_{market,       Tech,u} = NaN
w_{market,    Finance,ℓ} = NaN
w_{market,    Finance,u} = NaN
w_{market,  PetroChem,ℓ} = NaN
w_{market,  PetroChem,u} = NaN
w_{market,Automobile,ℓ} = NaN
w_{market,Automobile,u} = NaN
w_{eco,Y,ℓ} = NaN
w_{eco,Y,u} = NaN
```

```
z_{1} = NaN
z_{2} = NaN
z_{3} = NaN
z_{4} = NaN
z_{5} = NaN
z_{6} = NaN
z_{7} = NaN
z_{8} = NaN
z_{9} = NaN
z_{10} = NaN
z_{11} = NaN
z_{12} = NaN
z_{13} = NaN
z_{14} = NaN
z_{15} = NaN
y_{totinvest} = NaN
y_{risk,A,ℓ} = NaN
y_{risk,A,u} = NaN
y_{risk,B,ℓ} = NaN
y_{risk,B,u} = NaN
y_{risk,C,ℓ} = NaN
y_{risk,C,u} = NaN
y_{risk,D,ℓ} = NaN
y_{risk,D,u} = NaN
y_{market,       Tech,ℓ} = NaN
y_{market,       Tech,u} = NaN
y_{market,    Finance,ℓ} = NaN
y_{market,    Finance,u} = NaN
y_{market,  PetroChem,ℓ} = NaN
y_{market,  PetroChem,u} = NaN
y_{market,Automobile,ℓ} = NaN
y_{market,Automobile,u} = NaN
y_{eco,Y,ℓ} = NaN
y_{eco,Y,u} = NaN
```

Figure 1: Optimal solutions for portfolio optimization problem.