CSCI 5654, Spring 2023: Assignment 5

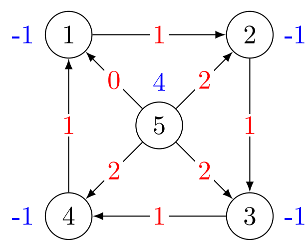
Assigned date: Fr 4/14/2023
Due date: We 5/3/2023
(Recommended to go through the homework before the Q&A of We 4/26)

Instructions: Please upload your HW as a PDF file.

Problem 1 (13 points): Network-flow problem

You can refer to the lectures or to Vanderbei's book, Chapter 14, for the terminology and theory of network-flow problems.

Consider the following network:



wherein the numbers (in blue) above the nodes are supplies (negative values represent demands) and the numbers (in red) shown on the arcs are unit shipping costs.

(a) Write the (primal) Linear Program corresponding to minimizing the shipping cost while meeting the demand. What are the decision variables? What are the constraints? What is the objective function?

(b) Write the dual of the Linear Program provided in (a).

Consider the spanning tree $\mathcal{T}$ given by the arcs $(5,1)$, $(5,2)$, $(5,3)$ and $(5,4)$.

(c) Give the primal solution (i.e., the value of the primal variables) associated to the spanning tree $\mathcal{T}$. Is this solution feasible? Give the associated cost.

(d) Give the dual solution (i.e., the value of the dual variables) associated to the spanning tree $\mathcal{T}$, setting $y_1 = 0$. Is this solution feasible? Give the associated cost.

(e) Using the primal network simplex method, what would be the leaving arc (only one possibility)? What would be the entering arc (only one possibility)? Give the spanning tree $\mathcal{T}'$ after the pivoting.

(f) Give the primal solution (i.e., the value of the primal variables) associated to the spanning tree $\mathcal{T}'$. Is this solution feasible? Give the associated cost.

(g) Give the dual solution (i.e., the value of the dual variables) associated to the spanning tree $\mathcal{T}'$, setting $y_1 = 0$. Is this solution feasible? Give the associated cost.

(h) Are the solutions given in (f) and (g) optimal? Justify your answer.

Problem 2 (7 points): Conic programming and inventory management

Consider the following optimization problem:

$$\begin{array}{ll} \min & 5x_1 + 2/x_1 + 3x_2 + 1/x_2 \\ \text{s.t.} & 2x_1 + 4x_2 \leq 3 \\ & x_1, x_2 > 0. \end{array} \qquad (1)$$

This problem appears for instance in optimal production and inventory management.[1] We will see, step by step, how to formulate this problem using conic programming.

Let $a > 0$ and consider the set $S = \{(s,t) : t > 0,\ s \geq a/t\}$.

(a) Show that $(s,t) \in S$ if and only if $\begin{bmatrix} s+t \\ s-t \\ 2\sqrt{a} \end{bmatrix} \in L_2^3$, where L_2^3 is the second-order cone in $\mathbb{R}^3$.

(b) By introducing two variables t_1 and t_2 satisfying $t_1 \geq 1/x_1$ and $t_2 \geq 1/x_2$, and using (a), formulate (1) as a Second-Order Cone Program with variables x_1, x_2, t_1, t_2. Your program should contain only linear expressions of the variables or constant terms, and linear or second-order cone inequalities on these expressions.

(c) Compute the dual of the conic program provided in (b). *Hint:* We remind that the dual of a conic program of the form

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & A_1 x \geq_{K_1} b_1 \\ & \vdots \\ & A_m x \geq_{K_m} b_m \end{array}$$

is given by

$$\begin{array}{ll} \max & b_1^\top y_1 + \ldots + b_m^\top y_m \\ \text{s.t.} & A_1^\top y_1 + \ldots + A_m^\top y_m = c \\ & y_1 \in K_1^*,\ \ldots,\ y_m \in K_m^*. \end{array}$$

(Bonus) [2] (challenging) Reduce the dual problem to a univariate optimization problem (i.e., the maximization of a function $f(y)$ with a single variable y). From this, obtain the optimal value of (1).

[1] See, e.g., `https://onlinelibrary.wiley.com/doi/10.1111/j.1540-5915.1989.tb01562.x`.

[2] For 1 point added to the final score of this assignment, with a ceil of 20 points for the final score.