Lyapunov Analysis

Lyapunov functions and barrier functions

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1 INTRODUCTION AND DEFINITIONS

Let n be a fixed positive integer. Let $\mathbb{T} \in \{\mathbb{Z}_{\geq 0}, \mathbb{R}_{\geq 0}\}$ be a *time domain* and $X \subseteq \mathbb{R}^n$ be a nonempty closed set called the *state space.*

Definition 1.1. A *dynamical system* (on $\mathbb{T}$ and X) is a continuous function $\phi : \mathbb{T} \times X \to X$ satisfying that (i) for all $x \in X$, $\phi(0, x) = x$ and (ii) for all $t_1, t_2 \in \mathbb{T}$, $t_2 > t_1$, and all $x \in X$, $\phi(t_2, x) = \phi(t_2 - t_1, \phi(t_1, x))$.

Definition 1.2. A dynamical system ϕ on $\mathbb{T} = \mathbb{R}_{\geq 0}$ is said to be *differentiable* if there is a function $f_\phi : X \to \mathbb{R}^n$ such that for all $x \in X$, $f_\phi(x) = \frac{\mathrm{d}}{\mathrm{d}t^+}\phi(t, x)\big|_{t=0} \doteq \lim_{t\to 0^+} \frac{\phi(t,x)-x}{t}$.

2 LYAPUNOV FUNCTIONS

Let ϕ be a dynamical system. In this section, we assume that $0 \in X$ and for all $t \in \mathbb{T}$, $\phi(t, 0) = 0$. In other words, 0 is a *fixed point* for ϕ.

Definition 2.1. 0 is a *stable fixed point* for ϕ if for every $\epsilon > 0$ there is $\delta > 0$ such that for all $x \in X$, $\|x\| \leq \delta$, and all $t \in \mathbb{T}$, $\|\phi(t, x)\| \leq \epsilon$.

Definition 2.2. 0 is an *asymptotically stable fixed point* for ϕ if (i) it is a stable fixed point, and (ii) for all $x \in X$, $\lim_{t\to\infty} \|\phi(t, x)\| = 0$.

Let us now introduce the concept of Lyapunov function.

Definition 2.3. A continuous function $V : X \to \mathbb{R}_{\geq 0}$ is a *candidate Lyapunov function* if (i) $V(0) = 0$, and for all $x \in X \setminus \{0\}$, $V(x) > 0$, and (ii) V is radially unbounded, meaning that $\lim_{r\to\infty} \min\{V(x) : x \in X,\ \|x\| \geq r\} = \infty$.

Definition 2.4. A candidate Lyapunov function $V : X \to \mathbb{R}_{\geq 0}$ is a *Lyapunov function* for ϕ if for every $t \in \mathbb{T}_{>0}$ and every $x \in X \setminus \{0\}$, $V(\phi(t, x)) < V(x)$.

THEOREM 2.5. *If ϕ has a Lyapunov function, then* 0 *is asymptotically stable for ϕ.*

PROOF. Let V be a Lyapunov function for ϕ. First, we show that 0 is stable. Therefore, fix $\epsilon > 0$. Let $c > 0$ be such that for all $x \in X$, $V(x) \leq c$ implies that $\|x\| \leq \epsilon$ (such c always exists since V is continuous, radially unbounded and nonzero on $X \setminus \{0\}$). Let $\delta > 0$ be such that for all $x \in X$, $\|x\| \leq \delta$ implies that $V(x) \leq c$ (such δ always exists since V is continuous and $V(0) = 0$). Now, for any $x \in X$, $\|x\| \leq \delta$, it holds that for all $t \in \mathbb{T}$, $V(\phi(t, x)) \leq V(x) \leq c$, so that $\|\phi(t, x)\| \leq \epsilon$. Thus 0 is stable.

Now, we show that 0 is attractive. Therefore, let $x \in X$. For a proof by contradiction, assume that $\liminf_{t\to\infty} \|\phi(t, x)\| > 0$. Since V is radially unbounded and since for all $t \in \mathbb{T}$, $V(\phi(t, x)) \leq V(x)$, it holds that $\limsup_{t\to\infty} \|\phi(t, x)\| < \infty$. Hence, there is $y \in X \setminus \{0\}$ and a diverging, increasing sequence $(t_i)_{i=0}^\infty$ such that $\lim_{i\to\infty} \phi(t_i, x) = y$. Without loss of generality, we may assume that for all $i \in \mathbb{N}$, $t_{i+1} > t_i + 1$. Since V and ϕ are continuous and V is a Lyapunov function, $V(\phi(1, y)) = \lim_{i\to\infty} V(\phi(t_i + 1, x)) \geq \lim_{i\to\infty} V(\phi(t_{i+1}, x)) = V(y)$. Hence $V(\phi(1, y)) \geq V(y)$, a contradiction with $y \neq 0$, concluding the proof. □

When V and ϕ are differentiable, a sufficient condition for V to be a Lyapunov function is given by looking at the derivatives of V and ϕ.

Proposition 2.6. *Let V be a differentiable candidate Lyapunov function and ϕ a differentiable dynamical system. Assume that for all $x \in X \setminus \{0\}$, $V'(x)f_\phi(x) < 0$. Then, V is a Lyapunov function for ϕ.*

Proof. Let $x \in X \setminus \{0\}$. For a proof by contradiction, assume there is $t \in \mathbb{R}_{>0}$ such that $V(\phi(t,x)) \geq V(x)$. Then, since $V \circ \phi$ is differentiable, by the mean value theorem, there is $s \in (0,t)$ such that $V'(\phi(s,x))f_\phi(\phi(s,x)) \geq 0$. Hence, by hypothesis, $\phi(s,x) = 0$, so that $V(\phi(s,x)) = 0$. By the same argument, it then follows that $V(\phi(t,x)) = 0$, so that $V(x) = 0$. This is a contradiction with $x \neq 0$. □

3 BARRIER FUNCTIONS

Let ϕ be a differentiable dynamical system. Let $I \subseteq X$ be nonempty closed set called the *initial set*, and $S \subseteq X$ be nonempty closed set called the *safe set*, with $I \subseteq S$.

Definition 3.1. S is *invariant* for (ϕ, I) if for every $x \in I$ and $t \in \mathbb{R}_{\geq 0}$, $\phi(t,x) \in S$.

Let us now introduce the concept of barrier function.

Definition 3.2. A differentiable function $B : X \to \mathbb{R}$ is a *candidate barrier function* for (I, S) if (i) for all $x \in I$, $B(x) \leq 0$, and (ii) for all $x \in X \setminus S$, $B(x) > 0$.

Definition 3.3. A candidate barrier function $B : X \to \mathbb{R}$ is a *barrier function* for (ϕ, I, S) if for every $x \in X$ such that $B(x) = 0$, it holds that $B'(x)f_\phi(x) < 0$.

Theorem 3.4. *If there is a barrier function for (ϕ, I, S), then S is invariant for (ϕ, I).*

Proof. Let B be a barrier function for (ϕ, I, S). Let $x \in I$. For a proof by contradiction, assume there is $t \in \mathbb{R}_{\geq 0}$ such that $\phi(t,x) \notin S$. Then, $B(\phi(t,x)) > 0$. Since $B(x) \leq 0$, this implies that there is $s \in [0,t)$ such that $B(\phi(s,x)) = 0$ and for all $u \in [s,t]$, $B(\phi(u,x)) \geq 0$. It follows that $B'(\phi(s,x))f_\phi(\phi(s,x)) \geq 0$, a contradiction with the hypothesis on B, concluding the proof. □