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Data-driven feedback stabilization of switched linear systems with probabilistic stability guarantees

Zheming Wang, Guillaume O. Berger and Raphaël M. Jungers

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Abstract— **This paper tackles the feedback stabilization of switched linear systems under arbitrary switching. We propose a data-driven approach which allows to compute a stabilizing static feedback using only a finite set of observations of trajectories without any knowledge of the dynamics. We assume that the switching signal is not observed, and as a consequence, we aim at solving a *uniform* stabilization problem in which the feedback is stabilizing for all possible switching sequences. In order to generalize the solution obtained from trajectories to the actual system, probabilistic guarantees are derived via geometric analysis in the spirit of scenario optimization. The performance of this approach is demonstrated on a few numerical examples.**

I. Introduction

Switched systems are typical hybrid dynamical systems which consist of a number of dynamics modes and a switching rule selecting the current mode. The jump from one mode to another often causes complicated hybrid behaviors resulting in significant challenges in stability analysis and control design, see [1], [2]. This paper focuses on the stabilization of switched linear systems. This problem has been an active area of research for many years, see, e.g., [3] and the references therein. In [4], [5], the (time) varying nature of dynamics is considered as uncertainty and uniform state feedback stabilization laws are proposed for all possible switching sequences. When both the control and the switching signal are accessible, exponential stabilization can be achieved for instance by using a piecewise quadratic control Lyapunov function [6]. In the presence of state and input constraints, stabilization of switched linear systems is also addressed under the framework of model predictive control [7]. However, these stabilization methods all require a model of the underlying switching system.

While there exist hybrid system identification techniques [8], identification of state-space models of switching systems is in general cumbersome and computationally demanding. More specially, identifying a switched linear system is NP-hard [9]. In recent years, data-driven analysis and control under the framework of black-box systems has received a lot of attention, see, e.g., [10]–[13]. For instance, probabilistic stability guarantees are provided in [11] for black-box switched linear systems, based merely on a finite number of observations of trajectories. Let us also mention that, although data-driven techniques for controlling linear systems already exist (see, e.g., [14]), they are not suitable for switched systems.

In this paper, we address the problem of stabilization of switched linear systems without any information on the model or the switching signal. As the switching is not within control, we need to design uniform stabilizing feedback for all the cases, similar to [4], [5]. More precisely, we compute a feedback controller and a common Lyapunov function for all the switching modes of the closed-loop system using a finite set of trajectories. The Lyapunov inequality leads to a finite set of bilinear matrix inequalities (BMI) and the stabilization problem becomes a BMI problem.

However, even though the data-based feedback controller stabilizes the trajectories obtained from the observations, it may not stabilize the actual system. In order to formally describe the properties of the controller, we derive probabilistic stability guarantees in the spirit of scenario optimization [15]–[17]. In this context, one trajectory can be considered as a scenario and the stabilization problem formulated based on a set of trajectories is a sampled problem. As our problem is non-convex, the convex chance-constrained theorems in [15] are not applicable. While chance-constrained theorems for nonlinear optimization problems also exist in [16]–[18], their probabilistic bounds rely on the knowledge of the essential set (which is basically the set of irremovable constraints). Identifying this set can be very expensive for general nonlinear problems, in particular for nonlinear semidefinite problems. Hence, the techniques in [16]–[18] are not suitable for our case which involves a large number of BMI constraints. Instead, in this paper, probabilistic guarantees on the computed controller are derived relying on the notion of covering number and packing number (see, e.g., Chapter 27 of [19]) and geometric analysis of the underlying problem. Similar probabilistic guarantees are also developed in [11], [20], [21] for autonomous systems. Note, however, that these guarantees require the optimality of the obtained solution, while our technique works with any feasible solution of the underlying optimization problem.

The rest of the paper is organized as follows. This section ends with the notation, followed by the next section on the review of preliminary results on stability of switched linear systems and the formulation of the stabilization problem. Section III presents the proposed data-driven stabilization approach with an alternating minimization algorithm and probabilistic stability analysis. In Section IV, we discuss some computational and practical issues of the proposed

The authors are with the ICTEAM Institute, UCLouvain, Louvain-la-Neuve,1348, Belgium. R Jungers is a FNRS honorary Research Associate. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 864017 - L2C. He is also supported by the Walloon Region and the Innoviris Foundation. G Berger is a FRIA/FNRS fellow. Email addresses: {zheming.wang,raphael.jungers,guillaume.berger}@uclouvain.be

approach. Numerical results are provided in Section V.

Notation. The non-negative integer set is denoted by $\mathbb{Z}^+$. For a square matrix Q, $Q \succ (\succeq)\ 0$ means that Q is symmetric and positive definite (semi-definite). $\mathbb{S}$ and $\mathbb{B}$ are the unit sphere and the unit (closed) ball in $\mathbb{R}^n$ respectively. $\mu(\cdot)$ denotes the uniform spherical measure on $\mathbb{S}$ with $\mu(\mathbb{S}) = 1$. For any matrix $P \succ 0$, we denote by $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ the largest and smallest eigenvalues of P respectively. Finally, given $x \in \mathbb{S}$ and $\theta \in [0, \pi/2]$, we let $\mathrm{Cap}(x, \theta) := \{v \in \mathbb{S} : |x^\top v| \geq \cos(\theta)\}$ be the symmetric spherical cap with direction x and angle θ.

II. Preliminaries and problem statement

We consider the following switched linear system

$$x(t+1) = A_{\sigma(t)}x(t) + Bu(t), \quad t \in \mathbb{Z}^+, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input and $\sigma(t) : \mathbb{Z}^+ \to \mathcal{M} := \{1, 2, \cdots, M\}$ is a time-dependent switching signal that indicates the current active mode of the system among M possible modes in $\mathcal{A} := \{A_1, A_2, \cdots, A_M\}$. In this paper, we consider the case in which the switching signal is changing arbitrarily and cannot be observed, i.e., the information on the switching signal is not available. Note that the input matrix B is constant. Our goal is to find a stabilizing state feedback $K \in \mathbb{R}^{m\times n}$ under arbitrary switching, i.e., the closed-loop system below is stable:

$$x(t+1) = (A_{\sigma(t)} + BK)x(t), \quad t \in \mathbb{Z}^+. \tag{2}$$

For notational convenience, let $\mathcal{A}_K := \{A_1 + BK, A_2 + BK, \cdots, A_M + BK\}$ for a given $K \in \mathbb{R}^{m\times n}$. The stability of System (2) under arbitrary switching can be described by the joint spectral radius (JSR) of the matrix set $\mathcal{A}_K$ defined by [22]

$$\rho(\mathcal{A}_K) := \lim_{k\to\infty} \max_{\boldsymbol{\sigma}(k)\in\mathcal{M}^k} \|\bar{\boldsymbol{A}}_{\boldsymbol{\sigma}(k)}(K)\|^{1/k} \tag{3}$$

where $\boldsymbol{\sigma}(k) := \{\sigma(0), \sigma(1), \cdots, \sigma(k-1)\}$ and $\bar{\boldsymbol{A}}_{\boldsymbol{\sigma}(k)}(K) = (A_{\sigma(k-1)} + BK)\cdots(A_{\sigma(1)} + BK)(A_{\sigma(0)} + BK)$. System (2) is asymptotically stable when $\rho(\mathcal{A}_K) < 1$. Hence, state feedback stabilization of System (1) amounts to finding a $K \in \mathbb{R}^{m\times n}$ such that $\rho(\mathcal{A}_K) < 1$. However, the computation of the JSR of a set of matrices is known to be a difficult problem except for some special cases, let alone its optimization in the context of control design. For this reason, we use tractable sufficient conditions for upper bounds on the JSR, see [22]. The following proposition provides a sufficient condition that can be computed via semidefinite programming [23].

Proposition 1 ([22, Prop. 2.8]): Consider the closed-loop matrices $\mathcal{A}_K$ for some state feedback $K \in \mathbb{R}^{m\times n}$. If there exist $\gamma \geq 0$ and $P \succ 0$ such that $A^\top PA \preceq \gamma^2 P, \forall A \in \mathcal{A}_K$, then $\rho(\mathcal{A}_K) \leq \gamma$.

From this proposition, we formulate the following nonlinear semidefinite optimization problem for stabilization of switched linear systems:

$$\min_{\gamma\geq 0, P, K} \gamma \tag{4a}$$
$$\text{s.t.} \quad (A + BK)^\top P(A + BK) \preceq \gamma^2 P, \ \forall A \in \mathcal{A} \tag{4b}$$
$$P \succ 0. \tag{4c}$$

Using the Schur complement [23] with $Q = P^{-1}$ and $Y = KQ$, the nonlinear constraints in (4) can be converted into linear matrix inequalities (LMI):

$$\min_{\gamma\geq 0, Q, Y} \gamma \tag{5a}$$
$$\text{s.t.} \quad \begin{pmatrix} \gamma^2 Q & QA^\top + Y^\top B^\top \\ AQ + BY & Q \end{pmatrix} \succeq 0, \ \forall A \in \mathcal{A} \tag{5b}$$
$$Q \succ 0. \tag{5c}$$

Such a transformation is widely used in stability analysis and control design, see, e.g., [24]. When the matrices $\mathcal{A}$ are known, Problem (5) can be efficiently solved via semidefinite programming and bisection on γ.

In this paper, we attempt to solve the stabilization problem of black-box switched linear systems (where the matrices $\mathcal{A}$ are unknown) in a data-driven fashion. To this end, we reformulate Problem (4) as a problem with infinite number of constraints below:

$$\gamma^* := \min_{\gamma\geq 0, P, K} \gamma \tag{6a}$$
$$\text{s.t.} \quad (Ax + BKx)^\top P(Ax + BKx) \leq \gamma^2 x^\top Px, \quad \forall A \in \mathcal{A}, \ \forall x \in \mathbb{S} \tag{6b}$$
$$P \succ 0. \tag{6c}$$

This problem is equivalent to Problem (4) thanks to the homogeneity of the closed-loop system in (2). As we will show later, the formulation in (6) allows us to develop model-free control design. Note that the transformation in (5) is no longer possible for Problem (6). The following assumption is needed.

Assumption 1: The state $x(t)$ can be fully observed for all $t \in \mathbb{Z}^+$, the input matrix B is time-invariant and known, and the number of modes (or an upper bound) is available.

The assumption that B is time-invariant is not restrictive in many applications, for instance, when the switching only occurs in some parameters of the dynamics. Such an assumption is often made in the literature, see, e.g., [4].

III. Main results

This section presents our model-free feedback stabilization method for black-box switched linear systems. We first formulate a sample-based stabilization problem, which consists of a set of bilinear matrix inequalities (BMI). Then, to solve this problem, we present an algorithm that generates feasible iterates. Finally, probabilistic guarantees on the obtained solution are provided via geometric analysis.

A. Sampled stabilization problem

For the model-free design, we sample a finite set of initial states and switching modes. More precisely, we randomly and uniformly generate N initial states on $\mathbb{S}$ and N modes in $\mathcal{M}$, which are denoted by $\omega_N := \{(x_i, \sigma_i) \in \mathbb{S} \times \mathcal{M} : i = 1, 2, \cdots, N\}$. From this random sampling, we observe the trajectories of the open-loop system of System (1) with $u = 0$ and obtain the data set $\{(x_i, A_{\sigma_i} x_i) : i = 1, 2, \cdots, N\}$, where $A_{\sigma_i} x_i$ is the successor of the initial state x_i. Note that the switching signal is not to be observed.

For the given data set ω_N (and $\{(x_i, A_{\sigma_i} x_i)\}_{i=1}^N$), we define the following *sampled* problem:

$$\min_{\gamma \geq 0, P \succ 0, K} \gamma \tag{7a}$$

$$\text{s.t.} \quad (A_\sigma x + BKx)^\top P (A_\sigma x + BKx) \leq \gamma^2 x^\top P x, \quad \forall (x, \sigma) \in \omega_N \tag{7b}$$

Using the Schur complement [23] and the homogeneity property (see the extended version in [25]), this problem can be equivalently written as the following BMI problem

$$\min_{\gamma \geq 0, P \succeq I, K} \gamma \tag{8a}$$

$$\text{s.t.} \quad \begin{pmatrix} \gamma^2 x^\top P x & (A_\sigma x + BKx)^\top P \\ P(A_\sigma x + BKx) & P \end{pmatrix} \succeq 0, \quad \forall (x, \sigma) \in \omega_N \tag{8b}$$

The advantage of this reformulation is that the nonlinear constraints in Problem (7) are decoupled into bilinear constraints, which allows to use an alternating algorithm as shown below.

B. An alternating algorithm

While quite a few algorithms and software packages are available for solving Problems (7) or (8), see, e.g., [26] and the references therein, we use an alternating minimization algorithm between P and K for its simple implementation. Thanks to the variable γ, feasibility is guaranteed at each iteration. Given a fixed P, we define:

$$\min_{\gamma \geq 0, K} \gamma \tag{9a}$$

$$\text{s.t.} \quad \begin{pmatrix} \gamma^2 x^\top P x & (A_\sigma x + BKx)^\top P \\ P(A_\sigma x + BKx) & P \end{pmatrix} \succeq 0, \quad \forall (x, \sigma) \in \omega_N \tag{9b}$$

When P is fixed in (8), the BMI constraints become LMI constraints and Problem (9) can be solved using convex optimization solvers [23]. Given a fixed K, we also define:

$$\min_{\gamma \geq 0, P \succeq I} \gamma \tag{10a}$$

$$\text{s.t.} \quad (A_\sigma x + BKx)^\top P (A_\sigma x + BKx) \leq \gamma^2 x^\top P x \quad \forall (x, \sigma) \in \omega_N \tag{10b}$$

This problem can be solved by bisection on γ with the solution of (9) being the initial guess. The overall procedure is summarized in Algorithm 1. Note that this alternating algorithm always terminates though it does not necessarily converge to a (local) optimum of Problem (8).

Algorithm 1 Alternating minimization for stabilization

Input: $\{(x_i, A_{\sigma_i} x_i)\}$, B and some tolerance $\epsilon_{tol} > 0$
Output: $\gamma(\omega_N), P(\omega_N)$, and $K(\omega_N)$

Initialization: Let $k \leftarrow 0$ and $P_k \leftarrow I_n$; Obtain K_k and γ_k from (9) with $P = P_k$;

1: Obtain P_{k+1} from (10) with $K = K_k$;
2: Obtain K_{k+1} and γ_{k+1} from (9) with $P = P_{k+1}$;
3: **if** $\|\gamma_{k+1} - \gamma_k\| < \epsilon_{tol}$ **then**
4: $\quad \gamma(\omega_N) \leftarrow \gamma_{k+1}$, $P(\omega_N) \leftarrow P_{k+1}$, $K(\omega_N) \leftarrow K_{k+1}$;
5: $\quad$ Terminate;
6: **else**
7: $\quad$ Let $k \leftarrow k + 1$ and go to Step 1.
8: **end if**

C. Probabilistic stability guarantees

We now derive formal stability guarantees on the solution obtained from Algorithm 1. Some definitions are needed. For any $\theta \in [0, \pi/2]$, we let $\delta(\theta)$ be the measure of the symmetric spherical cap with angle θ: i.e., $\delta(\theta) = \mu(\text{Cap}(x, \theta))$ for any $x \in \mathbb{S}$. The function δ is strictly increasing with θ, and thus we can define its inverse, denoted by δ^{-1}. In fact, it holds (see, e.g., [11]) that $\delta(\theta) = \mathcal{I}(\sin^2(\theta); \frac{n-1}{2}, \frac{1}{2})$, where $\mathcal{I}(x; a, b)$ is the regularized incomplete beta function defined as

$$\mathcal{I}(x; a, b) := \frac{\int_0^x t^{a-1}(1-t)^{b-1} dt}{\int_0^1 t^{a-1}(1-t)^{b-1} dt}; \tag{11}$$

see also Figure 1 for an illustration.

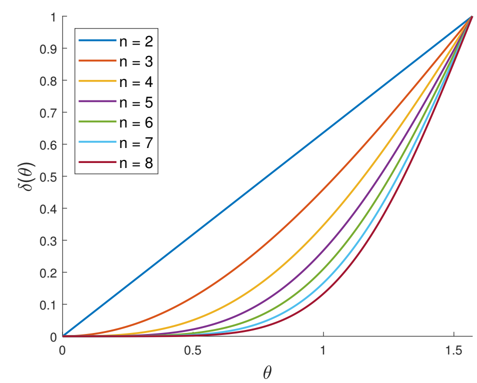

Fig. 1. Measure μ of the symmetric spherical cap $\text{Cap}(x, \theta)$ in $\mathbb{R}^n$ for different values of n.

Let us also recall the notions of covering and packing numbers, see Chapter 27 of [19] for details. We adapt the classic definitions to the unit sphere.

Definition 1: Given $\epsilon \in (0, 1)$, a set $Z \subset \mathbb{S}$ is called an ϵ-*covering* of $\mathbb{S}$ if, for any $x \in \mathbb{S}$, there exists $z \in Z$ such that $|z^\top x| \geq \cos(\theta)$ where $\theta = \delta^{-1}(\epsilon)$. The *covering number* $\mathcal{N}_c(\epsilon)$ is the minimal cardinality of an ϵ-covering of $\mathbb{S}$.

Definition 2: Given $\epsilon \in (0, 1)$, a set $Z \subset \mathbb{S}$ is called an ϵ-*packing* of $\mathbb{S}$ if, for any two $z, v \in Z$, $|z^\top v| < \cos(\theta)$ where $\theta = \delta^{-1}(\epsilon)$. The *packing number* $\mathcal{N}_p(\epsilon)$ is the maximal cardinality of an ϵ-packing of $\mathbb{S}$.

With these definitions, the following lemma is obtained.

Lemma 2: For any $\epsilon \in (0, 1)$,

$$\mathcal{N}_c(\epsilon) \le \mathcal{N}_p(\epsilon) \le \frac{1}{\delta(\frac{1}{2}\delta^{-1}(\epsilon))}. \tag{12}$$

Proof: The first inequality follows from the fact that any ϵ-packing with maximal cardinality is also an ϵ-covering. To prove the second inequality, let Z be the ϵ-packing with the maximal cardinality. Let $\theta = \delta^{-1}(\epsilon)$. From the definition of an ϵ-packing, the spherical caps $\{\mathrm{Cap}(z, \theta/2)\}_{z\in Z}$ are disjoint. Hence, $\sum_{z\in Z} \mu(\mathrm{Cap}(z, \theta/2)) \le 1$, which leads to the second inequality. ■

Remark 1: The definitions above are similar to those in [27], except that we consider *symmetric* spherical caps, to take into account the symmetry of the problem.

We then adapt the definition of ϵ-covering to the joint set $\mathbb{S} \times \mathcal{M}$ below.

Definition 3: Given $\epsilon \in (0, 1)$, a set $\omega \subset \mathbb{S} \times \mathcal{M}$ is called an *ϵ-covering* of $\mathbb{S} \times \mathcal{M}$ if, for any $(x, \sigma) \in \mathbb{S} \times \mathcal{M}$, there exists $z \in \mathbb{S}$ such that $(z, \sigma) \in \omega$ and $|z^\top x| \ge \cos(\theta)$ where $\theta = \delta^{-1}(\epsilon)$.

The following lemma shows probabilistic properties of the sample ω_N, which are needed for deriving formal guarantees on the controller.

Lemma 3: Given $N \in \mathbb{Z}^+$, let ω_N be independent and identically distributed (i.i.d) with respect to the uniform distribution $\mathbb{P}$ over $\mathbb{S} \times \mathcal{M}$. Then, given any $\epsilon \in (0, 1)$, with probability no smaller than $1 - \mathcal{B}(\epsilon; N)$, ω_N is a ϵ-covering of $\mathbb{S} \times \mathcal{M}$, where

$$\mathcal{B}(\epsilon; N) := \frac{M\left(1 - \frac{\delta\left(\frac{1}{2}\delta^{-1}(\epsilon)\right)}{M}\right)^N}{\delta(\frac{1}{4}\delta^{-1}(\epsilon))}. \tag{13}$$

Proof: Consider a maximal ϵ'-packing Z of $\mathbb{S}$ with $\epsilon' = \delta(\frac{1}{2}\delta^{-1}(\epsilon))$ and let $\theta = \delta^{-1}(\epsilon') = \frac{1}{2}\delta^{-1}(\epsilon)$. From the proof of Lemma 2, $\{\mathrm{Cap}(z, \theta)\}_{z\in Z}$ covers $\mathbb{S}$. Suppose ω_N is sampled randomly according to the uniform distribution, then the probability that each set in $\{\mathrm{Cap}(z, \theta)\}_{z\in Z}$ contains M points with M different modes is no smaller than $1 - \mathcal{N}_p(\epsilon')M(1 - \frac{\epsilon'}{M})^N \ge 1 - \mathcal{B}(\epsilon; N)$. When this happens, for any $(x, \sigma) \in \mathbb{S} \times \mathcal{M}$, there exists a pair $(z, \sigma) \in \omega_N$ such that $|x^\top z| \ge \cos(2\theta)$. This completes the proof. ■

The relation between $\rho(\mathcal{A}_{K(\omega_N)})$ and $\gamma(\omega_N)$ can be derived via geometric analysis, as shown below. Recall that $\rho(\mathcal{A}_{K(\omega_N)})$ is the JSR of the closed-loop system with $u = K(\omega_N)x$.

Lemma 4: Given $N \in \mathbb{Z}^+$ and $\omega_N \subset \mathbb{S} \times \mathcal{M}$, let $\gamma(\omega_N), P(\omega_N)$, and $K(\omega_N)$ be obtained from Algorithm 1. Suppose ω_N is an ϵ-covering of $\mathbb{S} \times \mathcal{M}$. Then,

$$\rho(\mathcal{A}_{K(\omega_N)}) \le \frac{\gamma(\omega_N)}{1 - \kappa(P(\omega_N))(1 - \cos(\delta^{-1}(\epsilon)))} \tag{14}$$

where $\kappa(P(\omega_N))$ is the condition number of $P(\omega_N)$, and with the convention that if $\kappa(P(\omega_N))(1-\cos(\delta^{-1}(\epsilon))) \ge 1$, then the right-hand side of (14) is infinite.

Proof: We only give a sketch of the proof due to the page limit and the detailed proof can be found in [25].

For notational convenience, we drop the argument in $P(\omega_N)$ and $K(\omega_N)$ in the proof. Consider the Cholesky decomposition of $P = L^\top L$, let

$$\tilde{\omega}_N := \left\{ \left(\frac{Lz}{\|Lz\|}, \sigma \right) : (z, \sigma) \in \omega_N \right\} \subset \mathbb{S} \times \mathcal{M}. \tag{15}$$

Step 1: We first show that if ω_N is an ϵ-covering of $\mathbb{S}\times\mathcal{M}$, then $\tilde{\omega}_N$ is an $\tilde{\epsilon}$-covering of $\mathbb{S} \times \mathcal{M}$ for some $\tilde{\epsilon} > 0$, that is, for any $(\tilde{x}, \sigma) \in \mathbb{S} \times \mathcal{M}$, we want to show that there exists $(\tilde{z}, \sigma) \in \tilde{\omega}_N$ such that $|\tilde{z}^\top \tilde{x}| \ge \cos(\tilde{\theta})$ where $\tilde{\theta} = \delta^{-1}(\tilde{\epsilon})$. Note that any $\tilde{x} \in \mathbb{S}$ can be uniquely expressed as $\tilde{x} = Lx/\|Lx\|$ for some $x \in \mathbb{S}$. Let $\tilde{x} = Lx/\|Lx\| \in \mathbb{S}$. Sine ω_N is an ϵ-covering of $\mathbb{S} \times \mathcal{M}$, from the definition, there exists $(z, \sigma) \in \omega_N$ such that $|x^\top z| \ge \cos(\theta)$ where $\theta = \delta^{-1}(\epsilon)$, which implies that $\|x-z\| \le \sqrt{2 - 2\cos(\theta)}$ or $\|x+z\| \le \sqrt{2 - 2\cos(\theta)}$. Now, let us look at the value $|\tilde{x}^\top \tilde{z}|$ where $\tilde{z} = Lz/\|Lz\| \in \tilde{\omega}_N$. Without loss of generality, we consider the case that $\|x - z\| \le \sqrt{2 - 2\cos(\theta)}$. Hence, via some manipulations, we get that [25, Lemma 4]

$$\frac{|(Lx)^\top Lz|}{\|Lx\|\|Lz\|} \ge 1 - \kappa(P)(1 - \cos(\theta)). \tag{16}$$

Hence, $\tilde{\omega}_N$ is a $\tilde{\epsilon}$-covering of $\mathbb{S} \times \mathcal{M}$ with $\tilde{\epsilon} = \delta(\tilde{\theta})$ and $\cos(\tilde{\theta}) = 1 - \kappa(P)(1 - \cos(\theta))$.

Step 2: Now, let us define:

$$\tilde{\omega}_N^\sigma := \{x : (x, \sigma) \in \tilde{\omega}_N\}, \; \forall\, \sigma \in \mathcal{M}. \tag{17}$$

Via geometric analysis, we get the following result [25, Lemma 4]:

$$\frac{\cos(\tilde{\theta})}{\gamma(\omega_N)} \tilde{A}_\sigma \mathbb{B} \subseteq \mathbb{B}, \forall \sigma \in \mathcal{M} \tag{18}$$

where $\tilde{A}_\sigma := LA_\sigma L^{-1} + LBKL^{-1}$. As a consequence, we obtain that, $\forall \sigma \in \mathcal{M}$,

$$(A_\sigma + BK)^\top P(A_\sigma + BK) \preceq \left(\frac{\gamma(\omega_N)}{\cos(\tilde{\theta})}\right)^2 P. \tag{19}$$

Finally, by combining (19) with Proposition 1, we get that $\frac{\gamma(\omega_N)}{\cos(\tilde{\theta})}$ is an upper bound on $\rho(\mathcal{A}_K)$, concluding the proof of the lemma. ■

Putting all the pieces together, we arrive at our main theorem.

Theorem 5: Given $N \in \mathbb{Z}^+$, let ω_N be i.i.d with respect to the uniform distribution $\mathbb{P}$ over $\mathbb{S} \times \mathcal{M}$. Suppose $\gamma(\omega_N), P(\omega_N)$, and $K(\omega_N)$ are obtained from Algorithm 1. Then, for any $\epsilon \in (0, 1)$, with probability no smaller than $1 - \mathcal{B}(\epsilon; N)$, the JSR of the closed-loop system (2) with $K = K(\omega_N)$ is bounded from above by $\gamma(\omega_N)/(1 - \kappa(P(\omega_N))(1-\cos(\delta^{-1}(\epsilon))))$, where $\mathcal{B}(\epsilon; N)$ is given in (13).

Proof: From Lemma 3, with probability no smaller than $1 - \mathcal{B}(\epsilon; N)$, ω_N is a ϵ-covering of $\mathbb{S} \times \mathcal{M}$. Combining this with Lemma 4, we obtain the statement. ■

Remark 2: The results above bear some similarities with the probabilistic stability guarantees in [11], [20], [21] which are concerned with autonomous systems, the major difference is that the bound in this paper is applicable for any feasible solution while [11], [20], [21] rely on the optimality of the solution.

IV. Computational and practical aspects

In this section, we discuss some computational and practical issues of our approach.

A. Input-state data and normalization

In some situations, the received data is a set of input-state data, i.e., $\{(x_i, u_i, x_i^+) : i = 1, 2, \cdots, N\}$ where $x_i^+ = A_{\sigma_i} x_i + Bu_i$ and u_i is the i^{th} input. As B is known, we can convert this data set into $\{(x_i, x_i^+ - Bu_i) : i = 1, 2, \cdots, N\}$. We can then apply our approach on this converted data set. Furthermore, the states may not lie on the unit sphere. While the solution of the sampled problem in (7) does not change from a theoretical point of view, we can use the scaled data $\{(x_i/\|x_i\|, (x_i^+ - Bu_i)/\|x_i\|) : i = 1, 2. \cdots, N\}$ to improve numerical stability. If the samples follow an isotropic Gaussian distribution centered at zero (with the covariance matrix being a scalar variance multiplied by the identity matrix) and are generated independently, the scaled points are uniformly distributed on the unit sphere and hence our probabilistic guarantees in Theorem 5 are still valid.

B. Sum of squares optimization

Quadratic stabilization of switched systems can be very restrictive. To reduce conservatism, we can use sum of squares (SOS) techniques, which have already been used in [28] to improve the bound on the JSR. In the framework of data-driven stability analysis, the application of SOS optimization has already been proved useful for the case of autonomous systems in [20]. Here, we want to show that SOS techniques are also applicable for the stabilization problem.

Let us first recall some definitions in SOS optimization [28]. Given $x \in \mathbb{R}^n$ and $d \in \mathbb{Z}^+$, let $x^{[d]}$ denote the d-lift of x which consists of all possible monomials of degree d, indexed by all the possible exponents α of degree d

$$x_\alpha^{[d]} = \sqrt{\alpha!}x^\alpha \tag{20}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\sum_{i=1}^n \alpha_i = d$ and $\alpha!$ denotes the multinomial coefficient

$$\alpha! := \frac{d!}{\alpha_1! \cdots \alpha_n!}. \tag{21}$$

The d-lift of a matrix $A \in \mathbb{R}^{n\times n}$ is defined as: $A^{[d]} : x^d \to (Ax)^{[d]}$. The following proposition provides a tighter bound for the JSR.

Proposition 6 ([22], Thm. 2.13): Consider the closed-loop matrices $\mathcal{A}_K$ for some state feedback $K \in \mathbb{R}^{m\times n}$. For any $d \in \mathbb{Z}^+ (d \geq 1)$, if there exist $\gamma \geq 0$ and $P \succ 0$ such that, $\forall\, A \in \mathcal{A}_\mathcal{K}$, $x \in \mathbb{S}$,

$$((Ax)^{[d]})^\top P(Ax)^{[d]} \leq \gamma^{2d}(x^{[d]})^\top P x^{[d]}, \tag{22}$$

where $P \in \mathbb{R}^{D\times D}$ with $D = \binom{n+d-1}{d}$, then $\rho(\mathcal{A}_K) \leq \gamma$.

In the model-free case, we formulate the following sampled problem using the given data set ω_N:

$$\min_{\gamma\geq 0, P\succ 0, K} \gamma \tag{23a}$$

$$\text{s.t.} \quad ((A_\sigma x + BKx)^{[d]})^\top P(A_\sigma x + BKx)^{[d]} \leq \gamma^2 (x^{[d]})^\top P x^{[d]},\ \forall\, (x, \sigma) \in \omega_N \tag{23b}$$

This becomes a polynomial optimization problem and is much more computationally demanding than Problem (8). For cases with moderate sizes, some methods and software toolboxes for solving such a problem are available [29]–[31]. To reduce the complexity, we can also use an alternating minimization scheme between P and K as shown in Algorithm 1 for the quadratic case. Similarly, once a feasible solution is obtained, we can also derive probabilistic guarantees by combing the results in Section III-C and [20].

V. Numerical experiments

Consider a switched linear system with 3 modes:

$$A_1 = \begin{pmatrix} 1.2 & 0.9 \\ -0.1 & 0.8 \end{pmatrix},\ A_2 = \begin{pmatrix} 1.8 & 3.2 \\ -0.5 & -0.16 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} -0.7 & -1.2 \\ 0.6 & 1.4 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

First, let $N = 1000$ and set the confidence level to $\mathcal{B}(\epsilon; N) = 0.01$. The corresponding value of ϵ is 0.0209, computed via bisection using (13). With this setting, the upper bound in (14) is valid with probability larger than 99%. For convenience, let

$$\overline{\gamma}(\omega_N) := \frac{\gamma(\omega_N)}{1 - \kappa(P(\omega_N))(1 - \cos(\delta^{-1}(\epsilon)))}.$$

We then generate ω_N according to the uniform distribution on $\mathbb{S} \times \mathcal{M}$ and apply Algorithm 1 with the tolerance being $\epsilon_{tol} = 0.1$. The obtained solution is:

$$\gamma(\omega_N) = 0.8365,\ K(\omega_N) = \begin{pmatrix} -0.2886 & -0.7086 \end{pmatrix},$$
$$P(\omega_N) = \begin{pmatrix} 4.3990 & 6.7572 \\ 6.7572 & 14.4331 \end{pmatrix}.$$

The bound in (14) is $\overline{\gamma}(\omega_N) = 0.8701$. To empirically verify the solution, we randomly generate a few trajectories of the closed-loop system with $K(\omega_N)$, see Figure 2.

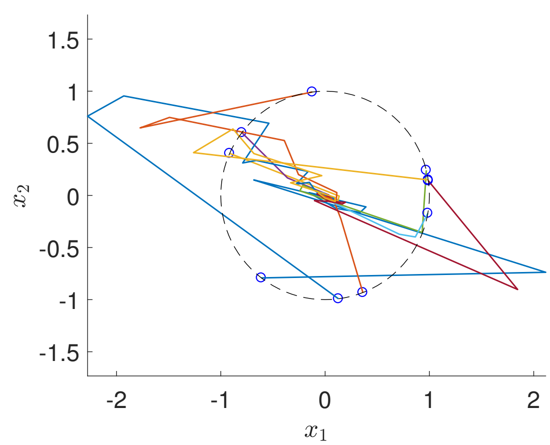

Fig. 2. Random trajectories of the closed-loop system: the blue circles are the initial states.

We then apply the proposed approach to higher dimensional examples. Again, the confidence level is set to 0.01, i.e., $\mathcal{B}(\epsilon; N) = 0.01$. We choose different values of N and compute the corresponding ϵ that satisfies $\mathcal{B}(\epsilon; N) = 0.01$. The input matrix B is set to be $\mathbf{1}_n$. The dynamics matrices $\mathcal{A}$ are generated randomly for different sizes. Note that these random examples may not be stabilizable by a static linear feedback. Hence, in the simulation, we only compute the upper bound $\overline{\gamma}(\omega_N)$ and compare it to the true solution γ^* defined in Problem (6). The results are shown in Figure 3.

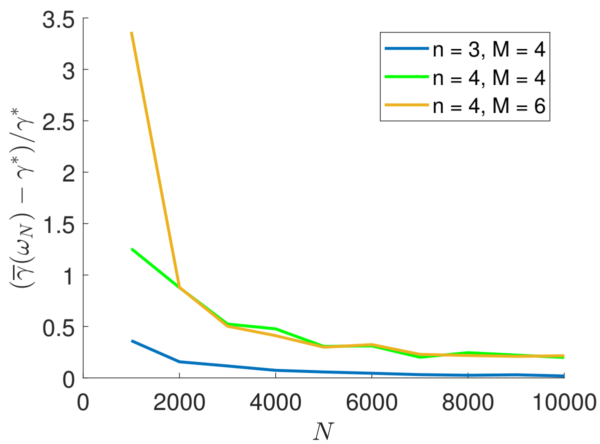

Fig. 3. Convergence of the sample-based solution to the true solution for systems of different dimensions and modes.

VI. Conclusions and Future Work

This paper proposes a data-driven approach for stabilization of black-box switched linear systems in which the dynamics matrices and the switching signal are unknown. The stabilization problem is formulated as a BMI problem using a finite number of trajectories. We then use an alternating minimization algorithm to solve this problem. While this data-based solution may not be a stabilizer for the actual system, probabilistic stability guarantees are provided using geometric analysis and the notions of covering number and packing number. In the future, we plan to extend this approach to nonlinear stabilizing feedback and output feedback stabilization.

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