

Data-driven invariant subspace identification for black-box switched linear systems

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Abstract—We present an algorithmic framework for the identification of candidate invariant subspaces for switched linear systems. Namely, the framework allows to compute an orthonormal basis in which the matrices of the system are close to block-triangular matrices, based on a finite set of observed one-step trajectories and with a priori confidence level. The link between the existence of an invariant subspace and a common block-triangularization of the system matrices is well known. Under some assumptions on the system, one can also infer the existence of an invariant subspace when the matrices are close to be block-triangular. Our approach relies on quadratic Lyapunov analysis and recent tools in scenario optimization. We present two applications of our results for problems of consensus and opinion dynamics; the first one allows to identify the disconnected components in a switching hidden network, while the second one identifies the stationary opinion vector of a switching gossip process with antagonistic interactions.

I. INTRODUCTION

Data-driven control systems have a rich and long history, encompassing system identification [1], controller synthesis [2]–[4], formal verification [5], etc. In recent years, we have seen a paradigm shift from *learning models* (system identification) to *learning solutions* (or certificates, like control barrier functions [6], abstractions [7], etc.). This has been enabled by the development of new techniques for learning, such as PAC (Probably Approximately Correct) Learning [8], [9] and scenario optimization [10], [11], which allow to provide solutions that are satisfactory with a high confidence level. This is particularly relevant for cyber-physical systems because system identification is in general very hard for these systems [1], and so are most control problems, even if the model of the system is known [12].

In this paper, we study the problem of identifying invariant subspaces for black-box switched linear systems in a data-driven way. Switched linear systems are systems described by a finite set of linear modes among which the system can switch over time. They constitute a paradigmatic class of cyber-physical systems, and appear naturally in many engineering applications or as abstractions of more complex systems [13]. Invariant subspaces are a central concept in linear system analysis; e.g., for safety verification, to perform dimensionality reduction in system analysis, or because

they contain important information about the system (as in consensus [14] or Markov chains [15]).

Our approach for subspace identification draws on scenario optimization [11]. In particular, the existence of a candidate invariant subspace is inferred from the existence of a degenerate Lyapunov function for the system. More precisely, the zero level-set of the Lyapunov function provides an orthonormal basis in which the matrices of the system are close (with bounded distance) to be block-triangular. Under some assumptions on the system, one can then infer the existence of an invariant subspace for the system; examples of such assumptions are discussed in the applications. By restricting to *quadratic* Lyapunov functions, the existence of such a function can be formulated as an SDP optimization problem. Because the system is black-box, the function has to be computed using a finite amount of data, and thus it is not guaranteed that the obtained quadratic function is a valid Lyapunov function for the system. However, by using results from scenario optimization, we can estimate with a priori-fixed confidence level, the probability of the set of one-step trajectories that are compatible with the obtained function. Using this estimation, we can bound the distance of the system matrices (in the appropriate orthonormal basis) to the set of block-triangular matrices. Note that a similar approach has been used in [16]–[19] for the data-driven computation of quadratic Lyapunov functions for stability analysis of switched linear systems. However, to the best of our knowledge, this work is the first one addressing the problem of common triangularization and invariant subspace identification for switched linear systems, without passing through a system identification phase.

We apply our technique on two problems in opinion dynamics. The first one is a problem of consensus over a switching hidden network [20]: by estimating the dimension of the dominant invariant subspace of the system, we are able to infer, from a finite set of observed one-step trajectories, the number of disconnected components in the network. The second application is a problem of opinion dynamics with antagonistic interactions [21]. The attracting opinion vector, if it exists, corresponds to a 1-dimensional invariant subspace of the system. Our technique allows us to identify, in a data-driven way, with high confidence level, such a subspace.

Outline. The paper is organized as follows. In Section II, we introduce the problem of interest. Several intermediate results follow, leading eventually to the main result of the paper in Subsection II-D, which allows to identify an orthonormal basis in which the system matrices are close to be block-triangular. Finally, in Section III, we present two

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applications of our framework for problems of consensus and opinion dynamics.

Notation. For vectors, $\|\cdot\|$ denotes the Euclidean norm, and for matrices, it denotes the spectral norm (largest singular value). $\|\cdot\|_F$ denotes the Frobenius norm. For a set $\mathcal{X} \subseteq \mathbb{R}^d$, \mathcal{X}^\perp denotes its orthogonal complement and $\text{conv}(\mathcal{X})$ its convex hull. \mathbb{S}^{d-1} is the unit Euclidean sphere in \mathbb{R}^d .

II. PROBLEM SETTING AND MAIN RESULT

A. Switched linear systems and invariant subspaces

We consider a discrete-time switched linear system, described by

$$\xi(t+1) = A_{\sigma(t)}\xi(t), \quad \xi(t) \in \mathbb{R}^n, \quad t \in \mathbb{N}, \quad (1)$$

where $\sigma : \mathbb{N} \rightarrow \mathcal{Q} := \{1, \dots, Q\}$ and for each $q \in \mathcal{Q}$, $A_q \in \mathbb{R}^{n \times n}$. The function σ is called the *switching signal*¹ of the system and specifies which *mode* (i.e., which transition matrix A_q) is used by the system at each time $t \in \mathbb{N}$.

An invariant subspace for System (1) is a (non-trivial) linear subspace $\mathcal{U} \subseteq \mathbb{R}^n$ satisfying that for every trajectory ξ of (1) with $\xi(0) \in \mathcal{U}$, it holds that for all $t \in \mathbb{N}$, $\xi(t) \in \mathcal{U}$. The existence of an invariant subspace is related to the common block-triangularization of the system matrices, as explained in the proposition below.

Proposition 1 [22, p. 12]: Let $\mathcal{U} \subseteq \mathbb{R}^n$ be a non-trivial linear subspace with dimension r , and let $U \in \mathbb{R}^{n \times n}$ be an orthogonal matrix whose first r columns are a basis of \mathcal{U} . The following are equivalent:

- 1) \mathcal{U} is invariant for System (1);
- 2) For each $q \in \mathcal{Q}$, there is $A_q^{(11)} \in \mathbb{R}^{r \times r}$, $A_q^{(12)} \in \mathbb{R}^{r \times (n-r)}$ and $A_q^{(22)} \in \mathbb{R}^{(n-r) \times (n-r)}$ such that

$$U^\top A_q U = \begin{bmatrix} A_q^{(11)} & A_q^{(12)} \\ 0 & A_q^{(22)} \end{bmatrix}. \quad (2)$$

Finding an invariant subspace from a finite set of trajectories is generally impossible if none of the trajectories starts inside the subspace. Therefore, we focus on finding a common “near” block-triangularization, that is, an orthogonal change of basis U in which the norm of the lower-left blocks — the would-be $A_q^{(21)}$ blocks in (2) — is bounded.

In our framework, the available data is a set of N one-step trajectories (x_i, y_i) , where $y_i = A_{q_i} x_i$ for some *unobserved* mode q_i ; i.e., we observe N starting points x_i and their respective images y_i by some latent mode q_i .

B. Quadratic Lyapunov approach

A common block-triangularization of System (1) can be obtained from a positive semi-definite (PSD) matrix P satisfying that for all $q \in \mathcal{Q}$, $A_q^\top P A_q \preceq \gamma^2 P$ for some $\gamma > 0$. Indeed, in that case, the kernel \mathcal{U} of P gives an invariant

¹In our framework (worst-case scenario analysis), the switching signal is an external input on which the user has no control, and the objective is to deduce properties of the systems that will be valid for every switching signal.

subspace.² Note that γ can be seen as a Lagrange multiplier arising in the \mathcal{S} -procedure [23].

When the matrices A_q are not available, we consider the data-driven version of the above approach. Namely, for a set $\Omega = \{(x_i, y_i)\}_{i=1}^N$ of observations and $\gamma > 0$, we consider the following optimization problem:

$$\begin{aligned} \min_P \quad & \|P\|_F^2 \\ \text{s.t.} \quad & y_i^\top P y_i \leq \gamma^2 x_i^\top P x_i, \quad \forall i \in \{1, \dots, N\}, \\ & P \succeq 0, \text{ trace}(P) = 1. \end{aligned} \quad (3)$$

We let $P_{\Omega, \gamma}^*$ be the optimal solution of (3), if it exists. In the following, we will assume that $\gamma > 0$ is fixed, and thus we will omit it in the notation.

The rationale of minimizing $\|P\|_F^2$ (which is equal to the sum of the squared eigenvalues of P) is to reduce the gap between the largest and smallest nonzero eigenvalues of P_{Ω}^* . In particular, if one defines the skewness of a PSD matrix P as in Definition 1 below, then minimizing $\|P\|_F$ induce its skewness to be small, which will be desirable in Theorems 4 and 5.

Definition 1: Let $P \in \mathbb{R}^{n \times n}$ be PSD with eigenvalues $\lambda_1 \geq \dots \geq \lambda_{n-r} > 0 = \lambda_{n-r+1} = \dots = \lambda_n$. We define the *skewness* of P by $\chi(P) = \lambda_{n-r}^{-1} \prod_{j=1}^{n-r} \lambda_j^{1/n}$.

We now establish the link between a solution of (3) and the existence of an orthogonal change of basis allowing to near block-triangularize the matrices of System (1). Therefore, define the set

$$\Psi_\Omega = \{(x, q) \in \mathbb{R}^n \times \mathcal{Q} : (A_q x)^\top P_{\Omega}^* (A_q x) \leq \gamma^2 x^\top P_{\Omega}^* x\} \quad (4)$$

of point–mode pairs compatible with P_{Ω}^* .

It holds that if Ψ_Ω is sufficiently covering the set $\mathbb{R}^n \times \mathcal{Q}$, then one can bound the norm of the lower-left blocks in a decomposition of System (1) akin to (2), where U is given by the kernel of P_{Ω}^* . This is stated in Theorem 2 below, but first we formalize the notion of “sufficiently covering” with the following concept of homogeneity (see also Figure 1 for an illustration).

Definition 2: Let $\mathcal{X} \subseteq \mathbb{R}^n$, $\epsilon > 0$ and $P_1, P_2 \in \mathbb{R}^{n \times n}$ be PSD. We say that \mathcal{X} is (ϵ, P_1, P_2) -homogeneous if for every $x \in \mathbb{R}^n$, there is $\mathcal{V} \subseteq \mathbb{R}^n$ such that $0 \in \text{conv}(\mathcal{V})$ and for every $v \in \mathcal{V}$, $v^\top P_1 x = 0$, $v^\top P_1 v \leq \epsilon^2 x^\top P_2 x$ and $x + v \in \mathcal{X}$.

Theorem 2: Consider System (1). Let $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ and $\epsilon \in (0, 1)$. Let P_{Ω}^* be the optimal solution of (3) and let Ψ_Ω be as in (4). Let $P_{\Omega}^* = U \Sigma^2 U^\top$ with $U \in \mathbb{R}^{n \times n}$ orthogonal, $\Sigma = \text{diag}(0_1, \dots, 0_r, \lambda_1, \dots, \lambda_{n-r})$ and $\lambda_1, \dots, \lambda_{n-r} > 0$. Let $\tilde{\Sigma} = \text{diag}(1_1, \dots, 1_r, \lambda_1, \dots, \lambda_{n-r})$ and $\tilde{P} = U \tilde{\Sigma}^2 U^\top$. For each $q \in \mathcal{Q}$, let $\mathcal{X}_q = \{x : (x, q) \in \Psi_\Omega\}$ and assume that \mathcal{X}_q is $(\epsilon, P_{\Omega}^*, \tilde{P})$ -homogeneous. Then, for each $q \in \mathcal{Q}$, there is $A_q^{(11)} \in \mathbb{R}^{r \times r}$, $A_q^{(12)} \in \mathbb{R}^{r \times (n-r)}$, $A_q^{(21)} \in \mathbb{R}^{(n-r) \times r}$ and $A_q^{(22)} \in \mathbb{R}^{(n-r) \times (n-r)}$ such that

$$U^\top A_q U = \begin{bmatrix} A_q^{(11)} & A_q^{(12)} \\ A_q^{(21)} & A_q^{(22)} \end{bmatrix}, \quad (5)$$

²This is easily seen, as for any $x \in \text{Ker}(P)$, $(A_q x)^\top P (A_q x) \leq \gamma^2 x^\top P x = 0$, so that $P A_q x = 0$ and thus $A_q x \in \text{Ker}(P)$.

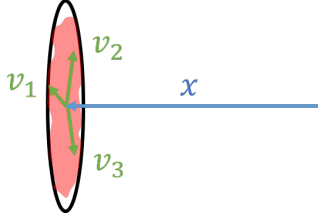


Fig. 1. (ϵ, P_1, P_2) -homogeneity. The red region is the intersection of \mathcal{X} with $x + \{v \in \mathbb{R}^n : v^\top P_1 x = 0, v^\top P_1 v \leq \epsilon^2 x^\top P_2 x\}$. \mathcal{X} is (ϵ, P_1, P_2) -homogeneous if for every x , there are vectors (here, e.g., v_1, v_2, v_3) whose convex hull contains x .

and $\|\Lambda A_q^{(21)}\| \leq \epsilon\gamma$ and $\|\Lambda A_q^{(22)} \Lambda^{-1}\| \leq \sqrt{1 + \epsilon^2}\gamma$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_{n-r})$.

Proof: See the extended version [24]. ■

C. Scenario approach and probabilistic guarantees

Theorem 2 states that if the set Ψ_Ω satisfies some homogeneity assumption, then we can obtain a common near block-triangularization of System (1) with a bound on the norm of the lower-left blocks. The difficulty is that it is in general impossible to compute the set Ψ_Ω from $\Omega = \{(x_i, y_i)\}_{i=1}^N$ if we do not have access to the matrices of the system. Nevertheless, if the point-mode pairs (x_i, q_i) generating Ω are sampled independently at random, then we can obtain *probabilistic guarantees* on the probability of Ψ_Ω . This is presented in the following theorem, which is the first main result of this subsection.

Theorem 3: Consider System (1) and let \mathbb{P} be a probability measure on $\mathbb{R}^n \times \mathcal{Q}$. Let $\beta \in (0, 1)$ be a confidence level and $N \in \mathbb{N}$ a number of samples. Let $\Omega = \{(x_i, y_i)\}_{i=1}^N$, where $y_i = A_{q_i} x_i$ and $\{(x_i, q_i)\}_{i=1}^N$ is sampled i.i.d. according to \mathbb{P} . Let P_Ω^* be the optimal solution of (3) and Ψ_Ω be as in (4). Then, with probability $1 - \beta$ on the sampling of $\{(x_i, q_i)\}_{i=1}^N$, it holds that $\mathbb{P}(\Psi_\Omega) \geq 1 - \bar{\epsilon}(\frac{n(n+1)}{2})$, where $\bar{\epsilon} : \{0, \dots, N\} \rightarrow [0, 1]$ is defined by $\bar{\epsilon}(N) = 1$ and for all $k \in \{0, \dots, N-1\}$, $\bar{\epsilon}(k)$ is the solution of the equation

$$\binom{N}{k} (1 - \bar{\epsilon}(k))^{N-k} - \frac{\beta}{N} \sum_{i=k}^{N-1} \binom{i}{k} (1 - \bar{\epsilon}(k))^{i-k} = 0. \quad (6)$$

Proof: The proof uses tools from scenario optimization [11] and convex optimization [25]. The proof is presented in the extended version [24]. ■

The dependence of $\bar{\epsilon}(\frac{n(n+1)}{2})$ on N for several values of β and n is presented in the extended version [24]. We see there that $1/\bar{\epsilon}(\frac{n(n+1)}{2})$ grows polynomially with N .

Remark 1: In some situations, it is useful to constrain the variable P in (3); e.g., to include prior information on P or to reduce the computation time by lowering the dimension of the problem. If the additional constraints are convex, then Theorem 3 applies in the very same way; the only thing that needs to be changed is to use $\bar{\epsilon}(d+1)$ instead of $\bar{\epsilon}(\frac{n(n+1)}{2})$, where d is the number of degrees of freedom of the variable P . Furthermore, if one does not want to fix γ a priori, but rather wants to find the smallest γ for which (3) is feasible (e.g., to make the bound on the norm of the lower blocks in

(5) as small as possible), then the problem becomes quasi-convex and one has to use $\bar{\epsilon}(2d+1)$ instead of $\bar{\epsilon}(d+1)$; see the extended version [24] for details. This approach is used for instance in the applications in Section III.

Theorem 3 states that with enough observations (N large), we can assume with high confidence that $\mathbb{P}(\Psi_\Omega)$ is close to 1. Now, we show that we can ensure homogeneity of the components of Ψ_Ω for each $q \in \mathcal{Q}$ from this property. Therefore, we assume in the following that \mathbb{P} is the *uniform spherical probability distribution* on $\mathbb{R}^n \times \mathcal{Q}$, denoted by \mathbb{P}_\circ and defined by: $\mathbb{P}_\circ(\mathcal{X} \times \{q\}) = \mu^{n-1}(\mathcal{X} \cap \mathbb{S}^{n-1})/Q$, where μ^{n-1} is the uniform spherical measure on \mathbb{S}^{n-1} (see the extended version [24] for details) and $Q = |\mathcal{Q}|$.

Theorem 4: Consider System (1). Let $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ and $\hat{\epsilon} \in (0, 1)$. Let P_Ω^* be the optimal solution of (3) and Ψ_Ω be as in (4). Assume that $\mathbb{P}_\circ(\Psi_\Omega) \geq 1 - \hat{\epsilon}$. Let $P_\Omega^* = U \Sigma^2 U^\top$, $\tilde{\Sigma}$ and \tilde{P} be as in Theorem 2. Let $\eta = Q \chi(P_\Omega^*) \hat{\epsilon}$, where $\chi(P_\Omega^*)$ is the skewness of P_Ω^* (see Definition 1). Assume that $\eta < \frac{1}{2}$ and let

$$\bar{s} = \sqrt{1 - \tilde{I}_{\frac{n-1}{2}, \frac{1}{2}}(2\eta)}, \quad (7)$$

where $\tilde{I}_{a,b}$ is the *inverse regularized beta function*.³ It holds that for each $q \in \mathcal{Q}$, \mathcal{X}_q is $(\epsilon, P_\Omega^*, \tilde{P})$ -homogeneous with $\epsilon = \sqrt{\bar{s}^2 - 1}$, where \mathcal{X}_q is as in Theorem 2.

Proof: See the extended version [24]. ■

The relation (7) between \bar{s} and η is presented in the extended version [24].

D. Main result

We are now able to state the main result of the paper.

Theorem 5 (Main result): Consider System (1). Let $\Omega = \{(x_i, y_i)\}_{i=1}^N$, where $y_i = A_{q_i} x_i$ and $\{(x_i, q_i)\}_{i=1}^N$ is sampled i.i.d. according to the uniform spherical probability distribution \mathbb{P}_\circ . Let P_Ω^* be the optimal solution of (3). Let $P_\Omega^* = U \Sigma^2 U^\top$ with $U \in \mathbb{R}^{n \times n}$ orthogonal, $\Sigma = \text{diag}(0_1, \dots, 0_r, \lambda_1, \dots, \lambda_{n-r})$ and $\lambda_1, \dots, \lambda_{n-r} > 0$. Let $\beta \in (0, 1)$ and $\eta = Q \chi(P_\Omega^*) \bar{\epsilon}(\frac{n(n+1)}{2})$, where $\bar{\epsilon}$ is as in (6) and $\chi(P_\Omega^*)$ is the skewness of P_Ω^* . Assume that $\eta < \frac{1}{2}$ and let \bar{s} be as in (7). Then, with probability $1 - \beta$ on the sampling of $\{(x_i, q_i)\}_{i=1}^N$, it holds that for each $q \in \mathcal{Q}$, there is $A_q^{(11)} \in \mathbb{R}^{r \times r}$, $A_q^{(12)} \in \mathbb{R}^{r \times (n-r)}$, $A_q^{(21)} \in \mathbb{R}^{(n-r) \times r}$ and $A_q^{(22)} \in \mathbb{R}^{(n-r) \times (n-r)}$ such that

$$U^\top A_q U = \begin{bmatrix} A_q^{(11)} & A_q^{(12)} \\ A_q^{(21)} & A_q^{(22)} \end{bmatrix},$$

and $\|\Lambda A_q^{(21)}\| \leq \epsilon\gamma$ and $\|\Lambda A_q^{(22)} \Lambda^{-1}\| \leq \sqrt{1 + \epsilon^2}\gamma$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_{n-r})$ and $\epsilon = \sqrt{\bar{s}^2 - 1}$.

Proof: Let $\hat{\epsilon} = \bar{\epsilon}(\frac{n(n+1)}{2})$. By Theorem 3, it holds that with probability $1 - \beta$ on the sampling, $\mathbb{P}_\circ(\Psi_\Omega) \geq 1 - \hat{\epsilon}$. Whenever this is the case, it holds by Theorem 4 that for each $q \in \mathcal{Q}$, \mathcal{X}_q is $(\epsilon, P_\Omega^*, \tilde{P})$ -homogeneous, where \mathcal{X}_q is as

³The inverse regularized beta function is the function $\tilde{I}_{a,b} : [0, 1] \rightarrow [0, 1]$ defined by $\tilde{I}_{a,b}(x) = y$ if and only if $\frac{\int_0^y t^{a-1} (1-t)^{b-1} dt}{\int_0^1 t^{a-1} (1-t)^{b-1} dt} = x$ [26].

in Theorem 2. Hence, we can apply Theorem 2, concluding the proof. ■

Theorem 5 says that if we sample enough points (N large), then, unless we are unlucky in our sampling (which happens with probability at most β), we can find an orthonormal basis in which we can bound the deviation of System (1) from an exact block-triangular form as in (2). A flowchart of the framework and the computation of the different quantities is presented in the extended version [24].

There are several sources of conservatism in the derivation of Theorem 5, the main ones are:

- The condition of homogeneity (Definition 2) involves every $x \in \mathbb{R}^n$, while in the proof of Theorem 2, we need the property of homogeneity to hold only for $x \in \text{Ker}(P_\Omega^*) \cup \text{Im}(P_\Omega^*)$. The reason we do not refine the definition of homogeneity is that uniform homogeneity is the best that can be deduced from the knowledge that $\mathbb{P}_\circ(\Psi_\Omega) \geq 1 - \bar{\epsilon}$ (Theorem 4).
- In the proof of Theorem 3, we use the upper bound $s^*(\psi) \leq d+1$ with $d = \frac{n(n+1)}{2} - 1$, while in practice the value of $s^*(\psi)$ can be much smaller. However, as noted in [11, p. 609], computing the exact value of $s^*(\psi)$ can be difficult as it amounts to look at all possible subsequences φ of ψ , whose number grows combinatorially with N . For this reason, we have considered only the upper bound in the numerical examples.
- In the proof of Theorem 4, it is actually shown that \mathcal{X}_q is $(\epsilon, \tilde{P}, \tilde{P})$ -homogeneous, while only the $(\epsilon, P_\Omega^*, \tilde{P})$ -homogeneity (which follows from $P_\Omega^* \preceq \tilde{P}$) is used. When P_Ω^* has many zero eigenvalues, the difference between P_Ω^* and \tilde{P} can be large. The reason we do not refine this result is that the derivation of Theorem 4 (see the extended version [24] for details) requires to take slices of a sphere, which can be obtained (after a change of coordinates) only from a non-degenerate ellipsoid.

III. APPLICATION TO CONSENSUS AND OPINION DYNAMICS

The link between the existence of an invariant subspace for the system and a common block-triangularization of the system was established in Proposition 1. However, we have not discussed this link when one has only a common *near* block-triangularization of the system as the one provided in Theorem 5. In fact, in general, it is not possible to ensure the existence of an invariant subspace from a non-exact block-triangularization of the system, without further information about the system. However, there are applications for which the available prior information about the system allows us to do so. We discuss two such applications in this section.

A. Consensus over switching hidden network

We consider the problem of consensus over a switching hidden network. The evolution of the value of the nodes in the network over time can be modeled as a switched linear system (1), where $\xi(t)$ is the state vector (n is the number of nodes) at time t and $A_{\sigma(t)}$ is the *unknown* interaction matrix at time t (see, e.g., [20]). For example, let us consider

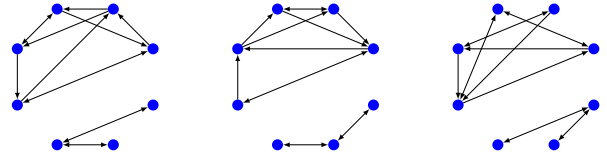


Fig. 2. Networks of nodes. A node j_1 is *adjacent* to a node j_2 if there is an edge from j_2 to j_1 in the network.

the system described by the networks $\{\mathcal{G}_q\}_{q \in \mathcal{Q}}$ depicted in Figure 2 ($n = 8$, $Q = 3$), and where the value of each node $j \in \{1, \dots, n\}$ is updated following the rule

$$\xi^{(j)}(t+1) = \frac{1}{1 + |\mathcal{N}_j^{\sigma(t)}|} \left(\xi^{(j)}(t) + \sum_{j' \in \mathcal{N}_j^{\sigma(t)}} \xi^{(j')}(t) \right), \quad (8)$$

with \mathcal{N}_j^q the set of nodes adjacent to j in \mathcal{G}_q .

In our setting, the networks are not known. The only things we know are the dimension n and the number of modes Q (or an upper bound on Q). The goal is to identify groups of nodes that *do not* interact with each other, that is, groups of nodes that are *disconnected* in the networks $\{\mathcal{G}_q\}_{q \in \mathcal{Q}}$. To address this problem with an invariant subspace approach, let us introduce the following result.

Proposition 6: Let \mathcal{G} be a simple directed graph with n nodes, and let $A \in \mathbb{R}^{n \times n}$ be the matrix of the associated linear system given by (8). Let $\mathcal{V} \subseteq \{1, \dots, n\}$ and let $u \in \{0, 1\}^n$ be the vector defined by, for all $j \in \{1, \dots, n\}$, $u^{(j)} = 1$ if and only if $j \in \mathcal{V}$. Then, \mathcal{V} and $\{1, \dots, n\} \setminus \mathcal{V}$ are disconnected if and only if $Au = u$.

Proof: The “only if” direction is clear. For the “if” direction, we proceed by contraposition. Therefore, assume that there is an edge from a node $j_1 \in \mathcal{V}$ to a node $j_2 \notin \mathcal{V}$ in \mathcal{G} . Then, $[Au]^{(j_1)} < u^{(j_1)}$, so that $Au \neq u$. Similarly, if there is an edge from a node $j_1 \notin \mathcal{V}$ to a node $j_2 \in \mathcal{V}$ in \mathcal{G} , then $[Au]^{(j_1)} > u^{(j_1)}$, so that $Au \neq u$. ■

To find an invariant subspace of the system given by Figure 2 and (8), we use the data-driven quadratic Lyapunov framework described in Section II. First, we compute the optimal solution of (3) with a relatively small sample set of one-step trajectories $\{(x_i, y_i)\}_{i=1}^{N_{\text{small}}}$, $N_{\text{small}} = 2000$, of the system. The kernel of the associated matrix P_{small}^* is given by

$$\text{Im} \begin{bmatrix} 0.26 & 0.25 & 0.25 & 0.22 & 0.29 & -0.48 & -0.49 & -0.46 \\ 0.38 & 0.33 & 0.39 & 0.31 & 0.42 & 0.32 & 0.33 & 0.33 \end{bmatrix}^\top,$$

which is close to the subspace $\mathcal{U} := \text{span}\{u_1, u_2\}$ where $u_1 = [1, 1, 1, 1, 1, 0, 0, 0]^\top$ and $u_2 = [0, 0, 0, 0, 0, 1, 1, 1]^\top$.⁴ Hence, we suspect that for all $q \in \mathcal{Q}$, the connected components of \mathcal{G}_q have the form $\mathcal{V} = \{1, \dots, n\}$, or $\mathcal{V} \subseteq \{1, \dots, 5\}$, or $\mathcal{V} \subseteq \{6, \dots, 8\}$.

To verify this hypothesis with high confidence, we solve (3) with a larger sample set of one-step trajectories ($N_{\text{large}} = 247122000$) and fixing $P = U^{(2)}(U^{(2)})^\top$ where the

⁴Here, we used, in a *compressed sensing* fashion, the prior information that the invariant subspace of interest, if it exists, is spanned by binary vectors.

columns of $U^{(2)} \in \mathbb{R}^{n \times (n-2)}$ are an orthonormal basis of \mathcal{U}^\perp . This time, we also try to minimize γ (see also Remark 1). This provides $\gamma^* = 0.69$. By fixing the confidence level to $\beta = 0.01$, it follows that $\bar{\epsilon}(1) = 1.86 \cdot 10^{-8}$.⁵ By Theorem 5, it then follows that the matrices of the system admit a decomposition (5), with $\epsilon = 0.122$, $\gamma = 0.69$, $\Lambda = I_{n-2}$ and $U = [\frac{u_1}{\sqrt{5}}, \frac{u_2}{\sqrt{3}}, U^{(2)}]$.

From this decomposition, we can now deduce that for each $q \in \mathcal{Q}$, the connected components of \mathcal{G}_q have the form $\mathcal{V} = \{1, \dots, n\}$, or $\mathcal{V} \subseteq \{1, \dots, 5\}$, or $\mathcal{V} \subseteq \{6, \dots, 8\}$. Indeed, assume there is a connected component $\mathcal{V} \subseteq \{1, \dots, n\}$ which has not this form, and let $u \in \{0, 1\}^n$ be defined by, for all $j \in \{1, \dots, n\}$, $u^{(j)} = 1$ if and only if $j \in \mathcal{V}$. Let $n_1 = |\mathcal{V} \cap \{1, \dots, 5\}|$ and $n_2 = |\mathcal{V} \cap \{6, \dots, 8\}|$. By assumption on \mathcal{V} , $n_1, n_2 \geq 1$ and $n_1 + n_2 \leq 7$. Now, let $u' = U^\top u = [(u'^{(1:2)})^\top, (u'^{(3:8)})^\top]^\top$, where $u'^{(1:2)} \in \mathbb{R}^2$ and $u'^{(3:8)} \in \mathbb{R}^6$. It holds that $u'^{(1:2)} = [\frac{n_1}{\sqrt{5}}, \frac{n_2}{\sqrt{3}}]$. Fix $q \in \mathcal{Q}$. If \mathcal{V} is a connected component of \mathcal{G}_q , it holds by Proposition 6 that $A_q u = u$, so that $A_q^{(21)} u'^{(1:2)} + A_q^{(22)} u'^{(3:8)} = u'^{(3:8)}$. By the above, it follows that

$$\|u'^{(1:2)}\| \geq \frac{1 - \sqrt{1 + \epsilon^2 \gamma}}{\epsilon \gamma} \|u'^{(3:8)}\| \geq 3.62 \|u'^{(3:8)}\|.$$

This implies that $\|u'^{(1:2)}\| \geq (1 + (1/3.62)^2)^{-1/2} \|u'\| \geq 0.96 \|u'\|$, so that $\frac{n_1^2}{5} + \frac{n_2^2}{3} \geq 0.96^2 (n_1 + n_2)$. The maximum of $(\frac{n_1^2}{5} + \frac{n_2^2}{3}) / (n_1 + n_2)$ is reached for $n_1 = 5$ and $n_2 = 2$, and equals 0.905. The latter is smaller than $0.96^2 = 0.9216$. Hence, it follows that u cannot satisfy $Au = u$, thus we can assert, with confidence level 99%, that \mathcal{V} is not a connected component of \mathcal{G}_q .

B. Opinion dynamics with antagonistic interactions

We consider a problem of opinion dynamics, where many agents (e.g., the population of a country) are divided into four groups (e.g., based on their political affinity), and these agents exchange opinions about topics. The interactions between the agents influence their opinion, and the relationships can be either friendly (in which case interaction of A with B increases the opinion of A with the opinion of B) or antagonistic (in which case interaction of A with B increases the opinion of A with the opposite of the opinion of B) [21]. More precisely, for this example, the interaction patterns between the different groups can take three different values ($Q = 3$), represented in Figure 3, and the value of the opinion of each group $j \in \{1, \dots, 4\}$ is updated as follows:

$$\xi^{(j)}(t+1) = \frac{1}{1 + n_j^q} \left(\xi^{(j)}(t) + \sum_{j' \neq j} \text{sign}_{j,j'}^{\sigma(t)} \xi^{(j')}(t) \right), \quad (9)$$

where n_j^q is the number of groups with which j interacts in the pattern q , $\text{sign}_{j,j'}^q = 1$ (resp. -1) if there is a friendly (resp. antagonistic) interaction of j with j' , and $\text{sign}_{j,j'}^q = 0$ if there is no interaction.

In our setting, we do not know what are the interaction patterns; we only know an upper bound on Q . But, we have

⁵We use $\bar{\epsilon}(1)$ because P is fixed, so that $d = 0$; see Remark 1.

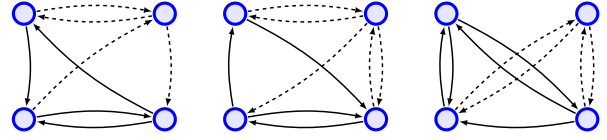


Fig. 3. Interaction patterns. A solid (dashed) arrow from A to B indicates an interaction of A with B with friendly (antagonistic) relationship.

access to the opinion vector (e.g., via polls or by counting the number of *likes* on social networks) at different time instants, which allows us to collect sample trajectories of the system. We are interested in deciding whether there is a *stable* opinion vector, that is, whether there is a normalized vector $u \in \mathbb{R}^4$, such that $A_q u = u$ for all $q \in \mathcal{Q}$ (where A_q is given by (9) with the interaction pattern q). Our results do not allow us to ensure the existence of such a stable opinion vector; however, we are able to decide what will be this stable vector if it exists.⁶

To identify such a potential stable vector, we apply the data-driven quadratic Lyapunov framework described in Section II. First, we compute the optimal solution of (3) with a relatively small sample set of one-step trajectories $\{(x_i, y_i)\}_{i=1}^{N_{\text{small}}}$, $N_{\text{small}} = 2000$, of the system. The associated optimal matrix satisfies $\text{Ker}(P_{\text{small}}^*) = \text{span}\{u\}$, where $u = [1, -1, 1, 1]^\top$. Hence, we suspect that u is a stable vector of the system. To verify this hypothesis with high confidence, we solve (3) with a larger sample set of one-step trajectories ($N_{\text{large}} = 8142000$) and fixing $P = U^{(2)}(U^{(2)})^\top$ where the columns of $U^{(2)} \in \mathbb{R}^{n \times (n-1)}$ are an orthonormal basis of $\{u\}^\perp$. This time, we also try to minimize γ (see also Remark 1). This provides $\gamma^* = 0.334$. By fixing the confidence level to $\beta = 0.01$, it follows that $\bar{\epsilon}(1) = 5.67 \cdot 10^{-7}$.⁷ By Theorem 5, it then follows that the matrices of the system admit a decomposition (5), with $\epsilon = 0.02$, $\gamma = 0.334$, $\Lambda = I_{n-1}$ and $U = [u, U^{(2)}]$.

From this decomposition, we deduce that if the system has a stable vector, then it must be close to u . Indeed, assume there is a normalized stable vector $v \in \mathbb{R}^4$ and let $v' = U^\top v = [v'^{(1)}, (v'^{(2:3)})^\top]^\top$, where $v'^{(1)} \in \mathbb{R}$ and $v'^{(2:3)} \in \mathbb{R}^3$. Fix $q \in \mathcal{Q}$. Since v is a stable vector, it holds that $A_q v = v$, so that $A_q^{(21)} v'^{(1)} + A_q^{(22)} v'^{(2:3)} = v'^{(2:3)}$. By the above, it follows that

$$|v'^{(1)}| \geq \frac{1 - \sqrt{1 + \epsilon^2 \gamma}}{\epsilon \gamma} \|v'^{(2:3)}\| \geq 99 \|v'^{(2:3)}\|.$$

This implies that $|v'^{(1)}| \geq (1 + (1/99)^2)^{-1/2} \geq 0.9999$. Hence, we finally deduce, with confidence level 99%, that any normalized stable vector v for the system, if it exists, must be close to u , in the sense that $\|u - \pm v\| \leq 0.0001$.

IV. CONCLUSIONS

We introduced a quadratic Lyapunov framework for data-driven identification of potential invariant subspaces of black-

⁶Once a potential stable vector is identified, one can apply further data-driven analysis to get a high confidence that this vector is indeed a stable one; however, for the sake of brevity and simplicity, we restrict here to the identification of potential stable vectors.

⁷We use $\bar{\epsilon}(1)$ because P is fixed, so that $d = 0$; see Remark 1.

box switched linear systems. This framework allows us to identify a potential invariant subspace for the system without knowing any mathematical model, by computing a quadratic Lyapunov function from a data-driven optimization program. We then leverage results from scenario optimization, quasi-convex optimization, and geometric analysis, to come up with probabilistic guarantees on the identification accuracy. We demonstrated the applicability of our framework on problems of consensus and opinion dynamics, for which the existence of an invariant subspace bears useful information about the system, thereby allowing us to study the dynamics of these systems in a data-driven way.

For further work, we plan to further investigate the possibility of adding constraints in the optimization program in order to leverage prior information about the system or the sampling (for instance, in the case of adaptive sampling), and also to fight the curse of dimensionality. Another angle of attack to fight the curse of dimensionality is to refine the result asserting the homogeneity of a set from its probability measure, or to relax the condition of homogeneity while keeping the property of a bound on the identification accuracy. Finally, we plan to provide other data-driven analysis results for the identification of invariant subspaces, for instance, regarding the minimal growth rate of the trajectories on the identified subspace, thereby allowing us to provide guarantees on the existence of an invariant subspace without relying on prior knowledge or assumptions as in the presented applications.

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