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Quantized Stabilization of Continuous-Time Switched Linear Systems

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***Abstract*—In this letter, we study the problem of stabilizing continuous-time switched linear systems via mode-dependent quantized state feedback. We derive a closed-form expression for the minimal information data rate from the coder to the controller necessary to achieve stabilization of the system. In particular, it is shown that the evaluation of the minimal data rate for stabilization reduces to the computation of the Lyapunov exponent of some lifted switched linear system, obtained from the original one by using tools from multilinear algebra, and thus can benefit from well-established algorithms for the computation of the Lyapunov exponent. In a second time, drawing upon this expression, we describe a practical coder–controller that stabilizes the system, and whose data rate can be as close as desired to the optimal data rate.**

***Index Terms*—Networked control systems, switched systems, quantized systems.**

I. Introduction

QUANTIZED control has been an important area of research in recent years. Many modern control systems (such as cyber-physical systems, IoT, etc.) involve spatially distributed components that communicate through a shared, digital communication network. Due to the digital nature of the network, all data must be quantized before transmission, resulting in quantization error that can have large negative effects on the performance of the control loop. Furthermore, in applications, the capacity of the network is often limited by cost, power, physical and/or security constraints. Consequently, a major challenge in the design of such networked systems is to determine the minimal communication data rate needed to achieve a given control objective. This fundamental question has attracted a lot of attention from the control community in the past decades, with great theoretical and practical advances; as surveyed in [3], [9], [19].

In this letter, we are interested in quantized control of *continuous-time Switched Linear Systems (SLSs)*. These systems are described by a finite set of linear modes among which the system can switch in time. As paradigmatic examples of hybrid and cyber-physical systems, they appear naturally in many engineering applications, or as abstractions of more complicated systems [4], [5], [14].

A popular setting in quantized control of switched and hybrid systems is the so-called *mode-dependent* quantized feedback [8], [11], [16], [18]. This setting assumes that the current mode of the system is always known by the coder–controller; see also Figure 1. (By contrast, "*mode-independent*" or "*sampled-mode*" quantized feedback requires that mode information is also quantized [6], [17].) Mode-dependent quantized feedback is motivated, for instance, by control problems involving networked switched systems with exogenous switching mechanism, or deterministically switched systems whose switching signal is not known at time of the coder–controller's and infrastructure's design (see also [2]), or to derive fundamental bounds on the data rate necessary for other quantized control settings. The mode-dependent quantized feedback setting has been studied mainly in the context of Markov Jump Linear Systems (discrete-time control-affine SLSs whose sequence of modes is dictated by a Markov chain). Constructive data rate bounds for their Mean Square Stabilization have been proposed, e.g., in [8], [16], [18], and an expression for the minimal data rate for Mean Square Stabilization, thought not computable in general, is derived in [11].

In this letter, we study mode-dependent quantized feedback control of continuous-time control-affine SLSs, and the control objective that is considered is their stabilization under *arbitrary* switching (see Figure 1). Our contribution is twofold. First, we provide a closed-form expression for the minimal data rate for stabilization of these systems. The minimal data rate is expressed as the Lyapunov exponent of some "lifted" system that represents the action of the original system on elements of volume (captured by algebraic constructions called *exterior algebras*). The computation of the minimal data rate can thereby benefit from well-established algorithms for the computation of the Lyapunov exponent [14]. Secondly, drawing on this expression, we describe a practical coder–controller that stabilizes the system and works whenever the channel data rate fits that bound. In summary, this letter combines several algebraic tools in control (exterior algebras, Lyapunov exponent) and shows that these concepts are key to the analysis

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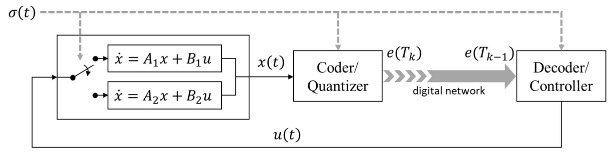

Fig. 1. Control of switched linear systems over digital communication networks with mode-dependent quantized feedback control loop.

of control-affine SLSs subject to data rate constraints. They allow for both an explicit theoretical characterization, and the practical computation, of optimal quantizing–controlling strategies.

Outline: The problem of interest is formulated precisely in Section II. In Section III, the closed-from expression for the minimal data rate for stabilization of SLSs, and a practical coder–controller that stabilizes the system, with a data rate as close as desired to the optimal bound, are presented. Finally, in Section IV, we demonstrate the applicability of our results on a numerical example.

Notation: For vectors, $\|\cdot\|$ denotes the Euclidean 2-norm, and for matrices it denotes the associated matrix norm (i.e., $\|M\|$ = largest singular value of M). $B(\xi, r)$ is the closed ball centered at $\xi \in \mathbb{R}^d$ with radius $r \geq 0$. $\lceil\cdot\rceil$ ($\lfloor\cdot\rfloor$) denotes the *ceil* (*floor*) operator. If $f : A \to B$, and $A' \subseteq A$, then $f|_{A'}$ denotes the restriction of f to the domain A'.

II. Preliminaries

A. Switched Linear Systems

Consider a continuous-time *Switched Linear System* (SLS) with affine control input:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad x(0) \in K, \quad t \geq 0, \tag{1}$$

where $\sigma(t) \in \Sigma := \{1, \ldots, N\}$ and $u(t) \in \mathbb{R}^c$, $A_i \in \mathbb{R}^{d\times d}$ and $B_i \in \mathbb{R}^{d\times c}$ for all $i \in \Sigma$, and $K \subseteq \mathbb{R}^d$ is a compact set with $0 \in \mathrm{int}(K)$. The function $\sigma : \mathbb{R}_{\geq 0} \to \Sigma$ is called the *switching signal* (or *s.s.* for short) and is assumed to be piecewise constant and right-continuous. For $\xi \in \mathbb{R}^d$ and $s \geq 0$, we denote by $x_{\sigma,u}(\cdot, s, \xi)$ the solution of (1) with s.s. σ, control input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^c$, and satisfying $x(s) = \xi$.[1]

We denote by $x_\sigma(\cdot, s, \xi) = x_{\sigma,0}(\cdot, s, \xi)$ the solution of the *open-loop* system (1) with s.s. σ and $x(s) = \xi$. For $t \geq s \geq 0$, the *state transition matrix* [13] from s to t of the open-loop system with s.s. σ is defined by

$$\Phi_\sigma(t, s) = e^{A_{\sigma(t_k)}(t-t_k)} \cdots e^{A_{\sigma(t_1)}(t_2-t_1)}e^{A_{\sigma(s)}(t_1-s)}, \tag{2}$$

where $t_1 < \ldots < t_k$ are the switching times of σ on $[s, t)$. Note that $x_\sigma(t, s, \xi) = \Phi_\sigma(t, s)\xi$ for all $\xi \in \mathbb{R}^d$.

We assume that the system is feedback stabilizable.

Definition 1: System (1) is said to be *feedback stabilizable* if there is a function $\varphi : \mathbb{R}^d \times \Sigma \to \mathbb{R}^c$ and constants $D \geq 0$ and $\mu > 0$ such that for every $\xi \in \mathbb{R}^d$ and s.s. σ, the feedback control input defined by $u(t) = \varphi(x(t), \sigma(t))$ satisfies

$$\|x_{\sigma,u}(t, 0, \xi)\| \leq D\|\xi\|\, e^{-\mu t} \qquad \forall\, t \geq 0. \tag{3}$$

[1]The linearity of the system implies that $x_{\sigma,u}(t, s, \xi) + x_{\sigma,v}(t, s, \eta) = x_{\sigma,u+v}(t, s, \xi + \eta)$. As a non-autonomous dynamical system, it also satisfies the *cocycle* property: $x_{\sigma,u}(t, r, \xi) = x_{\sigma,u}(t, s, x_{\sigma,u}(s, r, \xi))$.

B. Feedback Stabilization With Quantization and Data Rate Constraints

We investigate the problem of feedback stabilization of SLSs through digital networks with limited data rate. The situation is depicted in Figure 1. At specific transmission times, $0 \leq T_0 < T_1 < T_2 < \cdots$, a coder measures the state of the system, and is connected to a controller via a digital channel that can carry one discrete-valued symbol, selected from a finite coding alphabet $\mathcal{E}_k$, at each time T_k. A symbol sent at T_k is received by the controller at the latest at time T_{k+1}. Thus, at any time $t \in [T_{k+1}, T_{k+2})$, the controller has the symbols $e(T_0), \ldots, e(T_k)$ available and it generates an input $u(t)$ whose goal is to stabilize the system.

Let $(T_k)_{k\in\mathbb{N}}$ and $(\mathcal{E}_k)_{k\in\mathbb{N}}$ be the transmission times and the coding alphabets of the coder–controller. In general, those may depend on the switching signal; however, to not overload the notation, we will drop the dependence on the switching signal in the notation below. The symbol sent by the coder at time T_k is defined by

$$e(T_k) = \gamma_k(x(T_0), \ldots, x(T_k), \sigma|_{[0,T_k)}), \tag{4}$$

where $\gamma_k : (\mathbb{R}^d)^k \times \Sigma^{[0,T_k)} \to \mathcal{E}_k$ is the coder function at time T_k, and $x(\cdot)$ is the state of the system. The symbol $e(T_k)$ will be received by the controller at most at T_{k+1}. At any time $t \in [T_{k+1}, T_{k+2})$, the controller has thus the symbols $e(T_0), \ldots, e(T_k)$ available and it generates the input

$$u(t) = \zeta_t(e(T_0), \ldots, e(T_k), \sigma|_{[0,t)}), \tag{5}$$

where $\zeta_t : \mathcal{E}_0 \times \cdots \times \mathcal{E}_k \times \Sigma^{[0,t)} \to \mathbb{R}^c$ is the controller function at time t. Let $\gamma = (\gamma_k)_{k\in\mathbb{N}}$ and $\zeta = (\zeta_t)_{t\geq 0}$. The pair (γ, ζ) is called a *coder–controller*.

Definition 2: The coder–controller (γ, ζ) is said to *stabilize* (1) if the control input $u(\cdot)$ given by (4)–(5) satisfies

(a) *Exponential Convergence:* there are $C \geq 0$ and $\lambda > 0$ such that for every $\xi \in K$ and every s.s. σ, $\|x_{\sigma,u}(t, 0, \xi)\| \leq Ce^{-\lambda t}$ for all $t \geq 0$.

(b) *Lyapunov Stability:* for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $\xi \in B(0, \delta)$ and every s.s. σ, $\|x_{\sigma,u}(t, 0, \xi)\| \leq \varepsilon$ for all $t \geq 0$.

At each time T_k, the symbol $e(T_k)$ is transmitted via the communication network during the period $[T_k, T_{k+1})$. Using binary representation of the symbols, the minimal *data rate* (in bits/s) required for the network is thus given by

$$R(\gamma, \zeta) = \sup_\sigma \sup_{k\in\mathbb{N}} \frac{\lceil\log_2|\mathcal{E}_k|\rceil}{T_{k+1} - T_k}$$

where the first supremum is over all s.s. σ (remember that $(T_k)_{k\in\mathbb{N}}$ and $(\mathcal{E}_k)_{k\in\mathbb{N}}$ depend on σ).

Definition 3: The minimal data rate for stabilization of (1) is defined by[2]

$$R_{\mathrm{stab}}(A_\Sigma, B_\Sigma, K) = \inf_{(\gamma,\zeta)} R(\gamma, \zeta)$$

where the infimum is over all coders–controllers (γ, ζ) that stabilize the system.

[2]In formulas, it is convenient to identify system (1) by the triple (A_Σ, B_Σ, K) where $A_\Sigma = \{A_i\}_{i\in\Sigma}$ and $B_\Sigma = \{B_i\}_{i\in\Sigma}$.

III. Minimal Data Rate for Stabilization of SLSs

For control-affine LTI systems $\dot{x}(t) = Ax(t) + Bu(t)$, where (A, B) is stabilizable, it is well known that the minimal data rate for stabilization satisfies

$$R_{\text{stab}}(A, B, K) = \log_2(e) \sum_{\text{Re}(\lambda_i)>0} \text{Re}(\lambda_i), \tag{6}$$

where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of A. Moreover, for any data rate $R > R_{\text{stab}}(A, B, K)$, there is a *practical* coder–controller with data rate R that stabilizes the system.

In this section, we present a closed-form expression, similar to (6), for the minimal data rate for stabilization of SLSs. Moreover, drawing on this expression, we describe the implementation of a practical coder–controller that stabilizes the system and whose data rate can be arbitrarily close to $R_{\text{stab}}(A_\Sigma, B_\Sigma, K)$. The closed-form expression relies on the concepts of Lyapunov exponent [14] and of exterior powers of matrices [1]. For the sake of completeness, we remind below the definitions and properties of these concepts, relevant for this letter; see Sections III-A and III-B.

The results of this section are inspired from [2], where a closed-form expression, based on the *Joint Spectral Radius*,[3] for the minimal data rate required for state observation of *discrete-time* switched linear systems is presented.

A. Lyapunov Exponent

The *Lyapunov exponent* of the open-loop SLS (1), denoted by $\hat{\lambda}(A_\Sigma)$, is the smallest exponent α such that all trajectories of (1) in open-loop grow asymptotically slower that $e^{(\alpha+\varepsilon)t}$ for all $\varepsilon > 0$. Formally,

$$\hat{\lambda}(A_\Sigma) = \inf\{\alpha \in \mathbb{R} : \sup_{t\geq 0} e^{-\alpha t}\|x_\sigma(t, 0, \xi)\| < \infty \\ \forall \xi \in \mathbb{R}^d \text{ and s.s. } \sigma\}.$$

The Lyapunov exponent satisfies the following properties:

Proposition 1: Consider the open-loop SLS (1).

(i) For any $\alpha > \hat{\lambda}(A_\Sigma)$, there is $C \geq 0$ such that for every s.s. σ and $t \geq s \geq 0$, $\|\Phi_\sigma(t, s)\| \leq Ce^{\alpha(t-s)}$.

(ii) There is a switching signal σ such that $\limsup_{t\to\infty} e^{-\hat{\lambda}(A_\Sigma)t}\|\Phi_\sigma(t)\| > 0$.

Proof: The proof follows from [10, eq. (2)]. Due to space limitation, the details are omitted. ■

Note that for LTI systems $\dot{x}(t) = Ax(t)$, $\hat{\lambda}(A)$ is equal to the largest real part of the eigenvalues of A. It follows that $\hat{\lambda}(A_\Sigma)$ is at least equal to the largest real part of the eigenvalues of A_i for any $i \in \Sigma$ (use the s.s. $\sigma(\cdot) \equiv i$).

B. Exterior Powers of Matrices

Exterior algebras are algebraic constructions used to study the notions of areas, volumes, and their higher-dimensional analogues, in general vector spaces. In particular, exterior powers of linear operators are used to represent the action of linear operators on such elements of areas, volumes, etc. They can be defined in a coordinate-free fashion; see [1, Sec. 3.2.2]. However, in this letter, due to space limitation, we will restrict our attention to the exterior powers of *matrices*, which are themselves matrices and thus allow for a coordinate-based definition. To do this, let $\mathcal{I} = 2^{\{1,\ldots,d\}}$ be the set of all subsets, including $\emptyset$,[4] of $\{1, \ldots, d\}$. Let $A \in \mathbb{R}^{d\times d}$.

The *full-order exterior power* of A, denoted by $A^\wedge$, is the $2^d \times 2^d$ matrix whose entries are indexed by the elements of $\mathcal{I}$, and is defined for any $I, J \in \mathcal{I}$ by

$$A^\wedge_{I,J} = \begin{cases} 0 & \text{if } |I| \neq |J| \\ \det\big([A_{ij}]_{i\in I, j\in J}\big) & \text{otherwise.} \end{cases}$$

The 1*st-order exterior power* of A, denoted by $A^\odot$, is the $2^d \times 2^d$ matrix whose entries are indexed by the elements of $\mathcal{I}$, and is defined for any $I, J \in \mathcal{I}$ by

$$A^\odot_{I,J} = \begin{cases} 0 & \text{if } |I| \neq |J| \\ \sum_{k\in I} \det\Big([\tilde{A}^{(k)}_{ij}]_{i\in I, j\in J}\Big) & \text{otherwise,} \end{cases}$$

where $\tilde{A}^{(k)}$ is the $d \times d$ identity matrix with its kth column replaced by the kth column of A. (See also Section IV for a numerical example.)

The following proposition, whose proof can be found in [1, Sec. 3.2.3], summarizes all the properties of exterior powers of matrices that we will need in this letter.

Proposition 2: Let $A \in \mathbb{R}^{d\times d}$.

(i) $(\exp(A))^\wedge = \exp(A^\odot)$.

(ii) $\|A^\wedge\| = \prod_{i=1}^d \max\{\bar{\rho}_i, 1\}$ where $\bar{\rho}_1, \ldots, \bar{\rho}_d$ are the singular values of A.

(iii) The eigenvalues of $A^\odot$ are given by $\sum_{i\in I} \lambda_i$, $I \in \mathcal{I}$, where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of A.

Property (i) in Proposition 2 implies that $(\Phi_\sigma(\cdot, \cdot))^\wedge$ is the state transition matrix of the open-loop SLS (1) with set of matrices $\{A_i\}_{i\in\Sigma}$ replaced by $\{A_i^\odot\}_{i\in\Sigma}$.

C. Main Result

We are now able to present the main result of this letter, which combines the concepts of Lyapunov exponent and of exterior power into an efficiently computable formula for the minimal data rate for stabilization of SLSs.

Theorem 1: Assume that (1) is feedback stabilizable. The minimal data rate for stabilization of (1) satisfies

$$R_{\text{stab}}(A_\Sigma, B_\Sigma, K) = \log_2(e)\,\hat{\lambda}((A_\Sigma)^\odot), \tag{7}$$

where $(A_\Sigma)^\odot = \{A_i^\odot\}_{i\in\Sigma}$. Moreover, for any data rate $R > R_{\text{stab}}(A_\Sigma, B_\Sigma, K)$, there is a *practical* coder–controller with data rate R that stabilizes the system.

Note that by Proposition 2-(iii) and the comment below Proposition 1, (7) coincides with (6) when the SLS has only one mode. It also follows that the right-hand side term of (7) is always nonnegative since 0 is an eigenvalue of $A_i^\odot$ for any $i \in \Sigma$.

The proof that $R_{\text{stab}}(A_\Sigma, B_\Sigma, K) \geq \log_2(e)\,\hat{\lambda}((A_\Sigma)^\odot)$ is presented in Appendix A. The rest of the proof of Theorem 1 will follow from Section III-D where a practical coder–controller that stabilizes the system and works at any data rate $R > \log_2(e)\,\hat{\lambda}((A_\Sigma)^\odot)$ is described.

[3]The Joint Spectral Radius is a measure of stability of discrete-time SLSs.

[4]The following conventions will be useful when dealing with empty sets: an empty product of real numbers is equal to 1; an empty product of matrices is equal to the identity matrix; the determinant of an empty matrix is equal to 1; an empty sum is equal to 0.

Theorem 1 shows that the evaluation of $R_{\text{stab}}(A_\Sigma, B_\Sigma)$ can be reduced to the computation of the Lyapunov exponent of $A_\Sigma^\odot$, and thus can benefit from the numerous algorithms that have been proposed in the literature in the last decades to estimate the Lyapunov exponent of continuous-time SLSs; see [14] and references therein. The computation of the 1st-order exterior power of a matrix is straightforward from its definition, and thus $A_\Sigma^\odot$ can be computed in a systematic way. However, it should be noted that the dimension of the matrices in $A_\Sigma^\odot$ increases exponentially with the dimension of the system, and so will the complexity of approximating $\hat{\lambda}(A_\Sigma^\odot)$ (this is the curse of dimensionality). In this regard, let us mention that a simple algorithm-independent way to substantially speed up the approximation of $\hat{\lambda}(A_\Sigma^\odot)$, although not sufficient to beat the curse of dimensionality, is to use the fact that the matrices in $A_\Sigma^\odot$ are block diagonal, so that the computation of the Lyapunov exponent can be decoupled among the different diagonal blocks. Furthermore, there are cases for which the computation of $\hat{\lambda}(A_\Sigma)$ is straightforward. For instance, if A_Σ is a set of normal (or upper-, or lower-triangular) matrices, then the Lyapunov exponent is equal to the largest real part of the eigenvalues of the matrices A_i [14, Th. 2.41], [4, Proposition 2.2]. By combining these observations with the properties of the exterior powers of matrices (Proposition 2), we obtain very efficient ways to compute the minimal data rate for stabilization of SLSs with such sets of matrices; see also [2, Corollary 3.3] for similar results for the computation of the worst-case topological entropy of discrete-time SLSs.

D. Practical Coder–Controller

We describe the implementation of a practical coder–controller that stabilizes the system and operates at any data rate $R > \log_2(e)\,\hat{\lambda}((A_\Sigma)^\odot)$.

The following two lemmas are instrumental:

Lemma 1: Let $M \in \mathbb{R}^{d\times d}$ and $\alpha > 0$. Let $\mathcal{H} = MB(0,1)$. There is an m-points quantizer $Q(\cdot) : \mathbb{R}^d \to \mathcal{Q} \subseteq \mathbb{R}^d$ satisfying (i) $\|\xi - Q(\xi)\| \leq \alpha$ for all $\xi \in \mathcal{H}$, (ii) $Q(\xi) = 0$ if $\|\xi\| \leq \alpha/d^{1/2}$, and (iii)

$$m = |\mathcal{Q}| \leq \prod_{j=1}^{d} \left(2 \left[\!\!\left[\frac{d^{1/2}\bar{\rho}_i}{2\alpha} \right]\!\!\right] + 1 \right) =: \hat{m}_\alpha(M) \tag{8}$$

where $\bar{\rho}_1, \dots, \bar{\rho}_d$ are the singular values of M, and $[\![\cdot]\!]$ denotes the *rounding* (to the nearest integer) operator.

Proof: See Appendix B. ∎

Lemma 2: Consider system (1), and let $\alpha > 0$ and $R > \log_2(e)\,\hat{\lambda}((A_\Sigma)^\odot)$. There is $T^* \geq 0$ such that for every s.s. σ, and every $s \geq 0$ and $t \geq s + T^*$, it holds that $\hat{m}_\alpha(\Phi_\sigma(t,s)) \leq 2^{\lfloor R(t-s)\rfloor}$, where $\hat{m}_\alpha(\cdot)$ is as in (8).

Proof: See Appendix C. ∎

The coder–controller is defined as follows. (The reader may find useful to refer to Figure 2, where the different quantities appearing in the definitions are represented.)

1) Parameters: Fix $\alpha \in (0,1)$. Let $\varphi(\cdot,\cdot)$,[5] $D \geq 0$ and $\mu > 0$ be as in Definition 1. Let $T_* \geq 0$ be such that $De^{-\mu T_*} \leq \alpha$. Also, fix $r_0 \geq 0$ such that $K \subseteq B(0, r_0)$. Finally, fix $R > \log_2(e)\,\hat{\lambda}((A_\Sigma)^\odot)$, which will be the data rate of the coder–controller. For ease of notation, we also let $r_k = \alpha^k r_0$.

2) Coder and Controller Implementations: For the set of parameters defined above, the associated coder and controller are implemented by the algorithms in Figures 3 and 4.

[5] Without loss of generality, we may assume that $\varphi(0,i) = 0$ for all $i \in \Sigma$. This will be useful for the Lyapunov stability (see Appendix D).

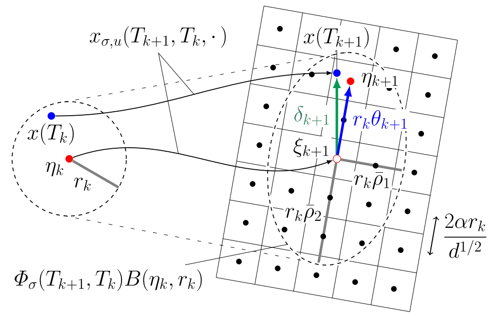

Fig. 2. The different quantities involved in the implementation of the coder–controller. The black points represent the quantized points, i.e., the set $\mathcal{Q}_k$ associated to $Q_k(\cdot)$, scaled by r_k and shifted by ξ_{k+1}.

Initialization: Let $k = 0$, $\eta_0 = 0 \in \mathbb{R}^d$, $T_0 = 0$, $e(T_0) =$ "empty symbol", and $\Delta_0 = T_*$.
Loop:

- **While** $t_{\text{real}} < T_k + \Delta_k$ or $\hat{m}_\alpha(\Phi_\sigma(t_{\text{real}}, T_k)) > 2^{\lfloor R(t_{\text{real}} - T_k)\rfloor}$:
  - Send symbol $e(T_k)$ to the controller.
- Let $k = k+1$ and $T_k = t_{\text{real}}$ (T_k is the first time for which the above while loop exits). Let Δ_k be the smallest real satisfying $\Delta_k \geq T_*$ and $\hat{m}_\alpha(\Phi_\sigma(T_k, T_{k-1})) \leq 2^{\lfloor R(\Delta_k)\rfloor}$. (In particular, it holds that $\Delta_k \leq T_k - T_{k-1}$.)
- Let $\xi_k = x_{\sigma,u}(T_k, T_{k-1}, \eta_{k-1})$ and $\delta_k = x(T_k) - \xi_k$.†
- Let $Q_k(\cdot)$ be the quantizer of Lemma 4 associated to $M := \Phi_\sigma(T_k, T_{k-1})$ and α. Let $\theta_k = Q_k(\delta_k / r_{k-1})$, and let $e(T_k)$ be a symbol that encodes θ_k.
- Finally, let $\eta_k = \xi_k + r_{k-1}\theta_k$ $(= x(T_k) - \delta_k + r_{k-1}\theta_k)$.

Fig. 3. **Coder implementation.** t_{real} denotes the current real time. †The input $u(\cdot)$ is determined by the controller (see Figure 4) based on the information sent by the coder, thus $u(\cdot)$ is also computable by the coder, and so is ξ_k; see also Remark 1. The knowledge of $x(T_k)$ comes from the fact that the coder may observe the current state of the system.

Initialization: Let $k = 0$, $\eta_0 = \xi_0 = 0 \in \mathbb{R}^d$, $T_0 = 0$, and $\Delta_0 = T_*$.
Loop:

- **While** $t_{\text{real}} < T_k + \Delta_k$ or $\hat{m}_\alpha(\Phi_\sigma(t_{\text{real}}, T_k)) > 2^{\lfloor R(t_{\text{real}} - T_k)\rfloor}$: *In parallel:*
  - Apply the following control input to the system:
    $$u(t_{\text{real}}) = \varphi(x_{\sigma,u}(t_{\text{real}}, T_k, \xi_k), \sigma(t_{\text{real}})). \tag{$\star$}$$
  - Receive the symbol $e(T_k)$.
- Let $k = k+1$ and $T_k = t_{\text{real}}$ (T_k is the first time for which the above while loop exits). Let Δ_k be the smallest real satisfying $\Delta_k \geq T_*$ and $\hat{m}_\alpha(\Phi_\sigma(T_k, T_{k-1})) \leq 2^{\lfloor R(\Delta_k)\rfloor}$.
- **If** $k \geq 2$: let θ_{k-1} be decoded‡ from $e(T_{k-1})$, and let $\eta_{k-1} = \xi_{k-1} + r_{k-2}\theta_{k-1}$.
- Let $\xi_k = x_{\sigma,u}(T_k, T_{k-1}, \eta_{k-1})$.

Fig. 4. **Controller implementation.** t_{real} denotes the current real time. ‡The controller can compute $\Phi_\sigma(T_{k-1}, T_{k-2})$ (see also Remark 1) and thus it is able to compute $Q_{k-1}(\cdot)$, and so obtain θ_{k-1} from $e(T_{k-1})$.

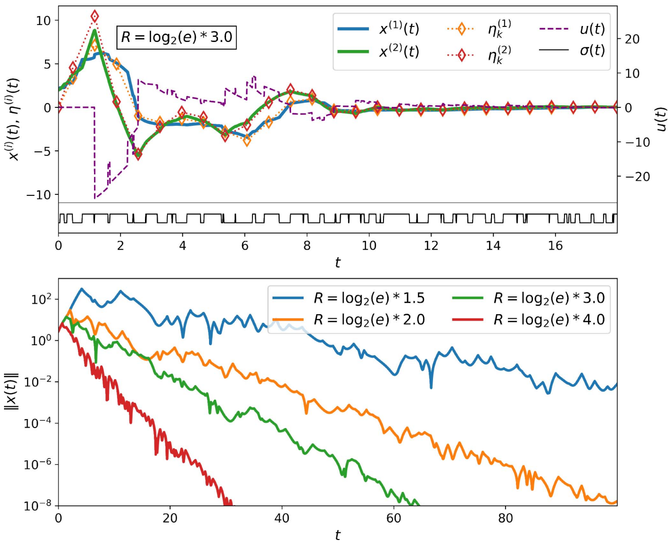

Fig. 5. Top: Evolution of $x(t)$ and $u(t)$ for a sample execution of the coder–controller with data rate $R = 3\log_2(e)$. The black curve below the plot represents the switching signal. The orange and red diamonds represent the value of η_k at the transmission times T_k. Bottom: Evolution of $\|x(t)\|$ for sample executions of the coder–controller with data rates $R \in \{1.5, 2, 3, 4\}\log_2(e)$.

The implementations deserves the following explanations. First, from their definition, the values of T_k, Δ_k, and ξ_k, are the same for the coder and the controller, for all $k \in \mathbb{N}$. Secondly, note that by Lemma 2, the while loop of the coder and controller always exits in finite time, and moreover, there is an upper bound T^* on $T_k - T_{k-1}$, independent of σ and k (this will be useful in the proof of the correctness of the coder–controller). Finally, by Lemma 1 and the definition of Δ_k, $Q_k(\cdot)$ is an m-points quantizer with $m \leq 2^{\lfloor R\Delta_k \rfloor}$. Hence, the symbol $e(T_k)$ can be transmitted, *with data rate* R, over the period $[T_k, T_{k+1})$ since $T_{k+1} \geq T_k + \Delta_k$.

Remark 1: Regarding the implementation, the coder and the controller know the switching times so that they can compute $\Phi_\sigma(\cdot, T_k)$, by (2). They can also compute ξ_k, $x_{\sigma,u}(\cdot, T_k, \eta_k)$ and ($\star$), by integrating auxiliary systems. In practice, these computations need not be carried out for every t_{real} (neither with infinite accuracy). For instance, we may fix a sampling period $\tau_s > 0$, and carry out the computations only at times $t_j = j\tau_s$, where τ_s is small enough to ensure that the "τ_s-sample-and-hold" (and noisy) version of $u(\cdot)$ in Definition 1 also stabilizes the system. The coder–controller can thus be implemented on a classical computer. Due to space limitation, the details are left for an extended version of this letter.

3) Correctness of the Coder–Controller: We have already shown (see Section III-D2) that the coder–controller is able to transmit the symbols with data rate R. The proof that the coder–controller stabilizes the system in the sense of Definition 2 is presented in Appendix D.

IV. Numerical Experiments

Consider the continuous-time SLS with $A_1 = \begin{bmatrix} 0.1 & 2.0 \\ 0.5 & 0.1 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.5 & 0.5 \\ 2.0 & 0.0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This system is stabilizable via the mode-dependent linear feedback law $u(t) = K_{\sigma(t)}x(t)$, with $K_1 = [\,-1.261 \quad -1.261\,]$ and $K_2 = [\,-2.5 \quad -0.823\,]$.

The 1-st-order exterior powers of A_1, A_2 are given by

$$A_1^{\odot} = \begin{bmatrix} 1 & & & \\ & 0.1 & 2.0 & \\ & 0.5 & 0.1 & \\ & & & 0.2 \end{bmatrix}, \quad A_2^{\odot} = \begin{bmatrix} 1 & & & \\ & -0.5 & 0.5 & \\ & 2.0 & 0.0 & \\ & & & -0.5 \end{bmatrix}$$

(where the indexes $I, J \in \mathcal{I}$ are ordered lexicographically.) We have used the results of [12], combined with [15], to estimate the Lyapunov exponent of $A_\Sigma^{\odot}$. This provided the interval $\hat{\lambda}(A_\Sigma^{\odot}) \in (1.21, 1.22)$. It follows that for any $R \geq 1.22\log_2(e)$, the coder–controller of Section III-D, with data rate R, stabilizes the system. A sample execution of the coder–controller applied on this system with data rate $R = 3\log_2(e)$ is represented in Figure 5-(top). A comparison of the stabilization rate of convergence for different values of the data rate of the coder–controller is presented in Figure 5-(bottom). As expected, we observe that the norm of $x(t)$ decreases more rapidly when the data rate is higher.

Appendix A
Proof That $R_{\text{stab}}(A_\Sigma, B_\Sigma, K) \geq \log_2(e)\,\hat{\lambda}((A_\Sigma)^{\odot})$

Fix a s.s. σ and $T > 0$. Let $\mathcal{U}_T$ be the set of input functions $u(\cdot)$ defined on $[0, T)$ by the coder–controller. Since, for a given s.s., the input $u(\cdot)$ depends only on the past received symbols, and because of the data rate constraint, the size of $\mathcal{U}_T$ is upper bounded by $2^{\lfloor RT \rfloor}$ where R is the data rate of the coder–controller. For each $u \in \mathcal{U}_T$, let Ξ_u be the set of points $\xi \in K$ with the following property: if the system starts from ξ, then the control input defined by the coder–controller, i.e., by (4)–(5), is equal to u on $[0, T)$. Clearly, $K = \bigcup_{u \in \mathcal{U}_T} \Xi_u$. Without loss of generality, we may assume that Ξ_u is Lebesgue measurable for all $u \in \mathcal{U}_T$. Finally, by linearity and by the stabilization property, for each $u \in \mathcal{U}_T$, it holds that if $\xi, \eta \in \Xi_u$, then $\|x_\sigma(T, \xi - \eta)\| \leq 2Ce^{-\lambda T}$ where C, λ are as in Definition 2-(a).

Now, for any $I \subseteq \{1, \ldots, d\}$ ($I \neq \emptyset$), let $\mathcal{V}_I \subseteq \mathbb{R}^d$ be the subspace spanned by $\{e_i\}_{i \in I}$ where e_i is the ith vector of the canonical basis of $\mathbb{R}^d$. For any set $\Lambda \subseteq \mathbb{R}^d$, let $\mathrm{vol}_I(\Lambda)$ be the $|I|$-dimensional Lebesgue volume of $\Lambda \cap \mathcal{V}_I$.

Now, fix $I, J \subseteq \{1, \ldots, d\}$, $|I| = |J| > 0$, and let $M \in \mathbb{R}^{|I| \times |I|}$ be the matrix obtained from $\Phi_\sigma(T, 0)$ by keeping only the rows with index in I and the columns with index in J. For each $u \in \mathcal{U}_T$, let $\Xi_{J,u} = \Xi_u \cap \mathcal{V}_J$. From the first paragraph, it holds that $\mathrm{vol}_I(\Phi_\sigma(T, 0)\Xi_{J,u}) \leq (2Ce^{-\lambda T})^{|I|}$. On the other hand, it holds that $\mathrm{vol}_I(\Phi_\sigma(T, 0)\Xi_{J,u}) = |\det(M)|\,\mathrm{vol}_J(\Xi_u) = |[(\Phi_\sigma(T, 0))^\wedge]_{I,J}|\,\mathrm{vol}_J(\Xi_u)$, where the second equality comes from the definition of the full-order exterior power. Thus, letting $\phi = |[(\Phi_\sigma(T, 0))^\wedge]_{I,J}|$, it holds that $\phi\,\mathrm{vol}_J(K) \leq \sum_{u \in \mathcal{U}_T} \phi\,\mathrm{vol}_J(\Xi_u) \leq 2^{\lfloor RT \rfloor}(2Ce^{-\lambda T})^{|I|}$. T is arbitrary, $\mathrm{vol}_J(K)$ is nonzero and independent of T, and C, λ are independent of σ. This implies that $2^{-RT}|[(\Phi_\sigma(T, 0))^\wedge]_{I,J}| \to 0$ as $T \to \infty$ for all s.s. σ. Since I, J are arbitrary, this implies that $2^{-RT}\|(\Phi_\sigma(T, 0))^\wedge\| \to 0$ as $T \to 0$ for all s.s. σ. Now, by Proposition 1-(ii) and the comment below Proposition 2, we deduce that $R \geq \log_2(e)\hat{\lambda}((A_\Sigma)^{\odot})$.

Appendix B
Proof of Lemma 1

Let USV^* be an SVD of M. Let $\beta = 2\alpha/d^{1/2}$, and for each $j \in \{1, \dots, d\}$, define $S_j = \{-[\![\bar{\rho}_j/\beta]\!], \dots, [\![\bar{\rho}_j/\beta]\!]\}$. It holds that $|S_j| = 2[\![\bar{\rho}_j/\beta]\!]+1$. Now, define $\mathcal{Q} = U(\beta S_1 \times \dots \times \beta S_d)$, and let $Q(\xi)$ be defined as the closest point in $\mathcal{Q}$ to ξ. Then, $Q(\cdot)$ satisfies (i)–(iii); due to space limitation, the details are left to the reader.

Appendix C
Proof of Lemma 2

First, we derive an upper bound on $\hat{m}_\alpha(M)$ in terms of the norm of its full-order exterior power $M^\wedge$. Let $\beta = d^{1/2}/(2\alpha)$. Note that for any $r \in \mathbb{R}$, it holds that $[\![r]\!] \le r + \frac{1}{2}$. Hence, $\hat{m}_\alpha(M) \le \prod_{i=1}^d (2\beta+2) \max\{\bar{\rho}_i, 1\} \le (2\beta+2)^d \|M^\wedge\|$, where the second inequality follows from Proposition 2-(ii).

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be such that $\hat{\lambda}((A_\Sigma)^\odot) < \lambda_1 < \lambda_2 < R/\log_2(e)$. Then, by Proposition 1-(i) and the comment below Proposition 2, there is $C \ge 0$ such that $\|(\Phi_\sigma(t,s))^\wedge\| \le Ce^{\lambda_1(t-s)}$ for all s.s. σ and $t \ge s \ge 0$. Thus, there is $T^* \ge 0$ such that $\hat{m}_\alpha(\Phi_\sigma(t,s)) \le (2\beta+2)^d \|(\Phi_\sigma(t,s))^\wedge\| \le \frac{1}{2}e^{\lambda_2(t-s)}$ for all s.s. σ and $s \ge 0$, $t \ge s + T^*$. The proof is complete by observing that $\frac{1}{2}e^{\lambda_2(t-s)} \le 2^{\lfloor R(t-s)\rfloor}$.

Appendix D
Proof That the Coder–Controller Stabilizes the System

We show that the coder–controller defined in Section III-D2 stabilizes the system in the sense of Definition 2. We proceed in steps. First, we show that for every $k \in \mathbb{N}$, it holds that $\|x(T_k) - \eta_k\| \le r_k$. This is obviously true for $k = 0$. Now, assume that it is true for some $k \in \mathbb{N}$, and observe that, by definition of δ_{k+1} and by linearity of the system, $\delta_{k+1} = x_\sigma(T_{k+1}, T_k, x(T_k) - \eta_k) = \Phi_\sigma(T_{k+1}, T_k)(x(T_k) - \eta_k)$. Thus, by the induction hypothesis, it holds that $\delta_{k+1} \in \Phi_\sigma(T_{k+1}, T_k)B(0, r_k)$. Hence, by definition of $Q_{k+1}(\cdot)$ and θ_{k+1}, we have that $\|\theta_{k+1} - \delta_{k+1}/r_k\| \le \alpha$. By definition of η_{k+1}, it follows that $\|x(T_{k+1}) - \eta_{k+1}\| \le \alpha r_k = r_{k+1}$. By induction, we conclude that this is satisfied for all $k \in \mathbb{N}$.

Secondly, we show that there is an upper bound on θ_k, independent of σ and $k \in \mathbb{N}$, and conclude that $\xi_k \to 0$ exponentially. The first claim comes from the observation that $\|\Phi_\sigma(T_k, T_{k-1})\| \le e^{LT^*}$, where $L = \max_{i\in\Sigma}\|A_i\|$ and T^* is the upper bound on $T_k - T_{k-1}$ discussed in §III-D2. Thus, we have that $\|\theta_k\| \le e^{LT^*} + \alpha$ for all $k \in \mathbb{N}$. For the second claim, observe that $\xi_{k+1} = x_{\sigma,u}(T_{k+1}, T_k, \eta_k) = x_\sigma(T_{k+1}, T_k, r_{k-1}\theta_k) + x_{\sigma,u}(T_{k+1}, T_k, \xi_k)$. Thus,

$$\begin{aligned}\|\xi_{k+1}\| &\le r_{k-1}e^{LT^*}\|\theta_k\| + De^{-\mu(T_{k+1}-T_k)}\|\xi_k\| \\ &\le C\alpha^k + \alpha\|\xi_k\|, \qquad C = r_0e^{LT^*}(e^{LT^*}+\alpha)/\alpha,\end{aligned}$$

(we used that $De^{-\mu(T_{k+1}-T_k)} \le De^{-\mu T_*} \le \alpha$ since $T_{k+1} - T_k \ge \Delta_k \ge T_*$; see Sections III-D1 and III-D2). From the above, it follows that $\xi_k \to 0$ exponentially as $k \to \infty$.

Finally, using the above results, we show that $x(t) \to 0$ exponentially. Indeed, from the definition of η_k, we have $\|x(T_k) - \xi_k\| \le \|x(T_k) - \eta_k\| + \|\eta_k - \xi_k\| \le r_k + r_{k-1}\|\theta_k\|$, which shows that $\|x(T_k) - \xi_k\| \to 0$ exponentially. Then, for $t \in [T_k, T_{k+1})$, we have from $x(t) = x_\sigma(t, T_k, x(T_k) - \xi_k) + x_{\sigma,u}(t, T_k, \xi_k)$, that

$$\begin{aligned}\|x(t)\| &\le e^{L(t-T_k)}\|x(T_k) - \xi_k\| + De^{-\mu(t-T_k)}\|\xi_k\| \\ &\le e^{LT^*}\|x(T_k) - \xi_k\| + D\|\xi_k\|,\end{aligned}$$

and thus, $x(t) \to 0$ exponentially as $t \to \infty$.

Summarizing, we have shown that the control input $u(\cdot)$ generated by the coder–controller satisfies the exponential convergence property (Definition 2). The proof that the origin is Lyapunov stable with this input is along the same lines as the proof of [7, Th. 1], and thus, omitted here. This concludes the proof that the coder–controller defined in §III-D2 stabilizes the system.

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