"Dominated splitting and quantization of hybrid systems : towards efficient control of cyber-physical systems and networked control systems"

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ABSTRACT

The goal of systems and control theory is to study natural phenomena (e.g., biological processes), technological devices (e.g., robots) or combinations of both (e.g., medical devices like pacemakers, etc.), and to design strategies to control them so that they behave in some intended way. For that, we rely on mathematical models describing the evolution of these phenomena/devices (called systems) and their reaction to external inputs. The challenge with modern systems is that these systems are becoming immensely complex. We think for instance to "cyber-physical systems", which result from the interaction of physical components and computerized components (e.g., self-driving cars where the dynamics of the car is governed by embedded or decentralized micro-controllers controlling it). These systems have become pervasive in our technological world, but they have also many non-standard characteristics (e.g., hybrid behavior, networked components), which preclude the use of classical control techniques and thus call for the development of new mathematical and algorithmic tools for their analysis and control. In this thesis, we study these two fundamental and challenging aspects of modern control systems (hybrid behavior and networked systems). For that, we leverage several tools from classical control theory and generalize them to switched or hybrid systems. In particular, we focus on the property that the dynamics of these systems can often be divided into several components, which grow at different speeds. This property, called "dominance", allows to study the convergence pr...

Ph.D. dissertation

Dominated splitting and quantization of hybrid systems

Towards efficient control of cyber-physical systems and networked control systems

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Thesis submitted in partial fulfillment of the requirements for the Degree of Doctor in Applied Sciences

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Dominated splitting and quantization of hybrid systems

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Abstract

The goal of systems and control theory is to study natural phenomena (e.g., biological processes), technological devices (e.g., robots) or combinations of both (e.g., medical devices like pacemakers, etc.), and to design strategies to control them so that they behave in some intended way. For that, we rely on mathematical models describing the evolution of these phenomena/devices (called systems) and their reaction to external inputs. The challenge with modern systems is that these systems are becoming immensely complex. We think for instance to "cyber-physical systems", which result from the interaction of physical components and computerized components (e.g., self-driving cars where the dynamics of the car is governed by embedded or decentralized micro-controllers controlling it). These systems have become pervasive in our technological world, but they have also many non-standard characteristics (e.g., hybrid behavior, networked components), which preclude the use of classical control techniques and thus call for the development of new mathematical and algorithmic tools for their analysis and control.

In this thesis, we study these two fundamental and challenging aspects of modern control systems (hybrid behavior and networked systems). For that, we leverage several tools from classical control theory and generalize them to switched or hybrid systems. In particular, we focus on the property that the dynamics of these systems can often be divided into several components, which grow at different speeds. This property, called "dominance", allows to study the convergence properties of these systems to low-dimensional attractors (with application for instance in population dynamics where the stability of the population composition amounts to the convergence of the system to a subspace of dimension one); or to study the stability of complex attractors for these systems (with application for instance in physics to study chaotic behaviors of electrical or meteorological systems). It is also highly relevant in networked control because dynamics that grow at different speeds generally do not require the same information flow (to be sent across the network) to be controlled satisfactorily. We provide both theoretical and algorithmic frameworks for the study of these questions for switched and networked systems, and we demonstrate their applicability on various numerical examples and concrete modern control problems.

Acknowledgments

A core and wonderful component of scientific research is about collaboration, exchanges and challenges of ideas. Therefore, I would like to express my gratitude to all people I had the pleasure to collaborate and discuss with during the four years of my PhD thesis.

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During the four years of the thesis, I had the privilege to work in a wonderful environment and to meet a lot of amazing people. Many thanks to my colleagues, Adrien, Antoine A., Benjamin, Benoît, Brieuc, Cécile, Charles, Émilie, Estelle, François G., François W., Guillaume O., Guillaume V.D., Julien C., Loïc, Lucas, Mahsa, Matteo, Matthew, Rémi, Sébastien C., Sébastien M., Virginie, Wei and Zheming, for having made these four years unforgettable. Also many thanks to the awesome academic and technical staff for their dedication in making the Applied Mathematics department such a unique, top-level, stimulating and enjoyable environment. In particular, I would like to sincerely thank Marie-Christine, Pascale and Étienne for their help with so many things regarding technical, administrative and organization issues.

Finally, I would like to express my great gratitude to my family, especially my parents, Dominique and Marc, and my sister, Victoria, for their support and encouragements, and to my friends for all the moments that gave me among others the energy to progress in the thesis.

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Contents

Preamble

Motivation: analysis and control of complex systems

A **system** is an entity or a group of interdependent entities that act according to a set of rules and are affected by their environment and/or external stimuli. Systems are ubiquitous in our technological world and in nature: for instance,

- in biology: populations of bacteria, plants, animals, etc., evolve according to some rules and are influenced by the environment, such as food supplies, climate conditions, etc.;
- in social science: the opinion or attitude of a social group can be seen as a system that evolves according to sociological and behavioral "rules", and is influenced by external stimuli, such as media, interactions with external people, etc.;
- in technologies: robots are devices including mechanical, electronic and computerized components that act as a whole governed by physical, electrical and algorithmic rules, and they are influenced by their environment and react to external inputs;

The goal of systems and control theory is to understand how these systems are working (**system analysis**), or to design control strategies, that is, defining a sequence of inputs for the system, so that the system acts in some predefined way (**control design**). To illustrate this, let us take the example of a platoon of self-driving cars. In most situations, it is important that the cars maintain a minimal safety distance between them; see also Figure 1 for an illustration. A control strategy to do that *could* be:

> "if the distance with the car in front is below some threshold: brake with maximal strength; if the distance with the car behind is below some threshold: put the acceleration at the maximum; otherwise: do nothing (except maintaining constant speed)".

However, this strategy is probably not the best one (in general such "bang-bang" strategies produce oscillations that rapidly become out of control). Another strategy could be to set the acceleration or braking intensity proportional to the gap between the actual and the ideal distance between cars. *Would this be a good strategy to solve the problem?* Mathematicians, scientists and engineers have been facing similar questions for ages (it goes back for instance to the work of James Watt on the regulation of steam machines). The best way we have found so far to answer these questions rigorously is to build a **mathematical model** of the system and to use mathematical tools to study it.

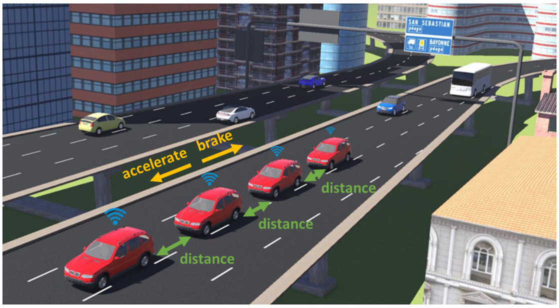


Figure 1: Platoon of self-driving cars (image adapted from Hu et al., 2020).

Systems and control engineers thus study mathematical models of systems to derive properties of the system or to design control strategies so that the system acts in some specified way. These models generally comprise **ordinary differential equations** to model the evolution of continuous variables (the velocity of the car is an example of continuous variable), **switching rules** to model the transitions of the system between different discrete states (the gear position of the car is an example of discrete variable), or **stochastic processes** to account for random phenomenons and unknown disturbances (the "patinage" of the car wheels is an example of random phenomenon).

Cyber-physical systems: a major rising challenge in modern control

Modern control systems are a big challenge for control theorists because these systems are becoming increasingly complex. We think for instance to **cyber-physical systems** (CPS), which are systems that include both physical and computerized components, with strong interactions between the two types of components. Self-driving cars are a good illustrative example of CPS: the physics of the system (velocity, acceleration, etc.) interacts with complex algorithms and software components implemented in decentralized or embedded micro-controllers.

Other examples of CPS include: Wireless Control Networks (where the controlling devices, which often consist of embedded and decentralized components, communicate through error-prone, physically constrained wireless channels), Smart Energy Grids (where local devices that produce or consume electricity have to computationally optimize the global behavior of the grid, taking into account physical and human constraints), autonomous robots, smart medical devices, and many others; see, e.g., Kim and Kumar (2012), Broy et al. (2012), Alur (2015), Lee and Seshia (2017) and Mitra (2021).

Systems including physical and computerized elements are not new, but it is only recently that it has become urgent to reason about these systems as a whole and not only as a collection of physical or cyber subsystems. To illustrate this, let us take the example of self-driving cars: for the control of these systems, one cannot dissociate the questions of control and communication, as issues in communication with the other cars will have a direct impact on the control of the car, and reversely the controlling devices of the car may trigger communication with the other cars when some conditions are met.

CPS have many **non-standard characteristics**, which preclude the use of classical control techniques; see, e.g., Figure 2 for some of these characteristics. Of particular importance,

- they have a **hybrid behavior**, which results from the interaction of discrete phenomenons (the "cyber" part) and continuous phenomenons (the "physical" part);
- they are subject to unknown or even **adversarial disturbances**, due to the presence of stochasticity and humans in the loop;
- they often involve spatially distributed components that communicate via a shared communication network (**networked systems**);
- they are subject to physical and **resource constraints**.

As a consequence, CPS are in general very hard to analyze, control, and design. Moreover, CPS are often involved in **safety-critical** applications (self-driving cars, medical devices, energy grids, etc.).

Several major advances in the understanding and design of these systems have been made in the last decades, allowing to provide workable solutions to a wide range of practical problems involving CPS, and contributing to the phenomenal expansion of CPS in the last years (coined as the *cyber-physical revolution*); see, e.g., Alur (2015), Lee and Seshia (2017) and Mitra (2021). This entailed the development of new techniques that are as interdisciplinary as the systems that they seek to control (namely, by combining tools from

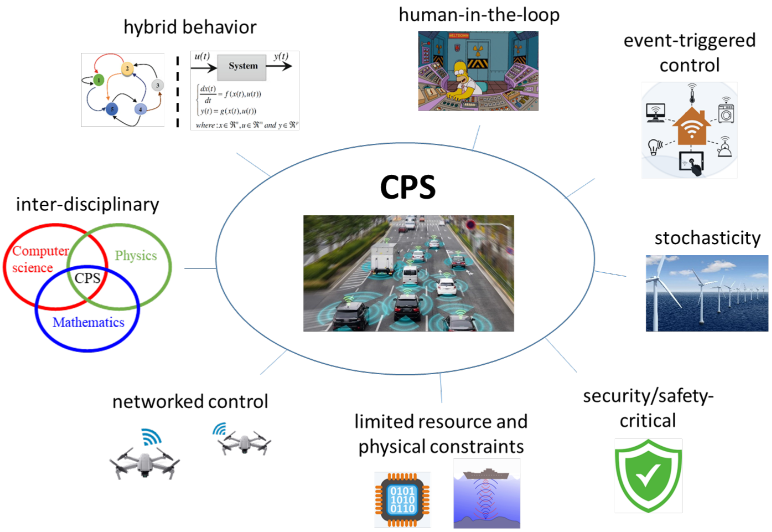


Figure 2: Some challenging aspects of CPS.

systems and control theory, computer science, information theory, physics, pure mathematics, etc.). However, despite the significant progress, CPS continue to be extremely challenging in terms of analysis and control, thereby calling for continued improvements and developments of techniques for their study and design.

Objectives and methodology: switched systems, worst-case analysis and quantization

The goal of this thesis is to provide mathematical and algorithmic tools for the analysis and control of complex systems, like cyber-physical systems.

Therefore, we consider several approaches to handle some of the challenging aspects of these systems.

For instance, we give a great deal of attention to **switched systems**, which are systems described by a finite set of "continuous" modes among which the system can switch over time. These systems thus combine continuous dynamics with discrete switching, thereby providing a paradigmatic class of hybrid systems. In particular, they appear naturally in a wide range of applications involving CPS, such as computer networks, digital power converters, viral dynamics, etc.; see, e.g., van der Schaft and Schumacher (2000), Liberzon (2003) and Jungers (2009).

Secondly, to study the problems of unknown or adversarial disturbances and safety requirements, we take the approach of **worst-case analysis**, whose goal is to provide formal guarantees that the system will satisfy the specifications in *all* situations (that is, it will work even in the "worst case"). In the case of switched systems, this generally amounts to derive properties of the system that are satisfied for all admissible sequences of modes of the system (that is, whatever the way the system is switching among its different continuous modes).

Finally, to study the challenges posed by networked systems and resource limitations, we focus on the **quantized control** of these systems, whose goal is to study the impact of quantization and limited information for the observation and control of these systems. Quantization is the process of representing continuous variables with quantities from a finite set, thereby introducing round-off or quantization errors. This occurs for instance when such variables are measured with digital sensors or are converted into finite-bits numbers for handling by a computer; see, e.g., Figure 3 for an illustration. Quantization and limited information flow (which occurs unavoidably when the information is carried by communication networks) can have important negative effects on the performance of control strategies for networked systems. The goal of quantized control is to study these effects and determine requirements on the quantization and information flow to ensure proper working the system.

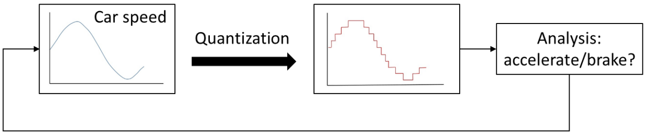


Figure 3: Quantization in self-driving car control.

Contributions: dominated splitting and quantization of hybrid systems

In this thesis, we draw on the observation that the dynamics of systems can often be separated into two or more components, such that each component grows with a different speed. For instance, in opinion dynamics, it often happens that some opinions are shared more slowly than other ones, so that these opinions eventually disappear, supplanted by faster-spread ideas. This property, called sometimes *fast and slow modes*[1] *separation*, has proved useful in a wide range of contexts: for instance, to identify **dominant trends** in the behavior of systems; or to study the convergence of systems beyond classical convergence to fixed points (as we will explain below in this introduction); also, in quantized control, the notion of separation of the dynamics plays a central

[1]The term "mode" should be understood here has "component of the dynamics"; it does not refer to the continuous modes of switched systems.

role, since dynamics that grow at different speeds generally require different levels of quantization.

> In this thesis, we study the property of separation of the dynamics for hybrid systems. We provide mathematical and algorithmic frameworks for the theoretical and practical analysis of the property, and we study several applications of it, in particular, in quantized control.

For that, we leverage several tools from applied mathematics. Let us present in this introduction two important such tools: namely, the concepts of hyperbolicity and exterior algebras.

Hyperbolicity is a central concept in systems theory. This concept accounts for the fact that the "linearized" dynamics of the system can be split into two components: a unstable component and a stable component. The unstable component has a dynamics that grows exponentially fast, while the stable component has a dynamics that converges exponentially fast to zero.

The property of stable–unstable dynamics separation allows to study a wide range of complex behaviors of systems, such as the existence of *strange attractors* for such systems.[2] This happens for instance when the unstable dynamics is visible only "from inside" the attractor (thereby allowing for rich behavior inside the attractor, while "from outside", the trajectories of the system seem to simply converge to the attractor); see also Figure 4 for an illustration. In this sense, the notion of hyperbolicity connects with the well-known notions of **safety** and **reachability** in control theory, which study questions like "will the trajectories of the system starting in some safe region stay in that safe region?" or "will the trajectories reach some target set eventually?". In the case of hyperbolic dynamics, the behavior on the safe or target set needs not be a stable or convergent dynamics, as opposed to some other classical notions of stability in systems and control theory.

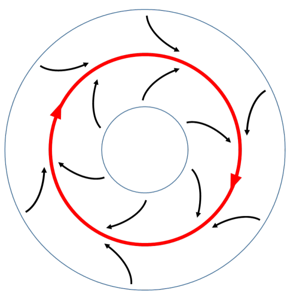


Figure 4: Example of hyperbolic system (see Example 2.9 in Subsection 2.3.3). The dynamics is unstable in the direction of the "circular flow" (red line), and is stable in the transversal direction. Consequently, the trajectories converge to the red set, but the behavior of the system on the red set is unstable.

[2] The notion of hyperbolicity emerged namely from the study of chaotic behaviors in engineering problems; see, e.g., Cartwright and Littlewood (1945).

The property of hyperbolicity has also proved useful in quantized control; see, e.g., Kawan (2013). To illustrate this, let us take the example of the property of **shadowing**, which accounts for the fact that "almost-trajectories" of hyperbolic systems are close to genuine trajectories of the system. Almost-trajectories occur for instance when one applies to the system a control input that is a quantized version of the nominal or intended input; the property of hyperbolicity tells us that if the quantization is sufficiently fine, the trajectories of the quantized-controlled system will be close to the trajectories of the intended-controlled system. In this sense, the property of shadowing connects with the one of **incremental stability**, which is a well-known property in systems and control theory and describes systems whose trajectories starting from nearby initial conditions converge to each other.

Exterior algebras are a very useful tool in mathematics. They allow to generalize the notions of length, area and volume in higher dimensions (that is, beyond 3D spaces). By analyzing the rate at which such elements of "higher-dimensional volume" grow under the action of a system, one can infer the growth rate of different components of the dynamics of the system. For instance, hyperbolicity implies that all elements of volume with some dimension grow faster (under the action of the system) than all elements of volume with higher dimensions; see Figure 5 for an illustration. We will also use this property for the quantized control of hybrid systems: as already mentioned before, dynamics that grow at different speeds generally require different levels of quantization; by combining this observation with the properties of exterior algebras, we analyze the "worst-case minimal data rate" for observation and control of switched linear systems.

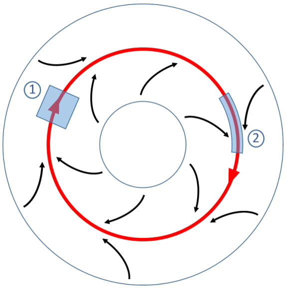


Figure 5: Hyperbolic system of Figure 4. The element of area (rectangle "1") is mapped by the system to region "2". The rectangle is expanded in the direction parallel to the red line and is contracted in the radial direction.

Let us now detail our contributions. The research that we conducted regarding the study of the separation of dynamics for hybrid systems can be divided into two main topics that we present below: namely, the dominance analysis of hybrid systems and the quantized control of these systems.

Dominance analysis of hybrid systems

We introduce the notion of **dominance** to study the property of having a separation of the dynamics for hybrid systems. This notion generalizes the one of hyperbolicity by considering splittings of the dynamics into dominant and dominated components, but without requiring that the dominated dynamics converges exponentially to zero (hyperbolicity can thus be seen as a special case of dominance; see below for some details).

We study the property of dominance, from theoretical and algorithmic points of view, for switched linear systems (a paradigmatic class of hybrid systems) and for smooth nonlinear systems.

First, we study this property for <u>switched linear systems</u>. Therefore, we introduce the notion of **dominated splitting** to describe the separation of the dynamics for these systems. We show that the trajectories of a switched linear system with a dominated splitting converge to a low-dimensional time-varying attractor; this convergence property can be seen as the "switched-linear-system" counterpart of the property of hyperbolic systems to admit attractors with an unstable dynamics; it can also be seen as the property of incremental stability (discussed above in this introduction) when the system is considered as acting on subspaces of fixed dimension (i.e., acting on the *projective space* or on the *Grassmannian manifold*).

As for the algorithmic aspects, we provide a geometric characterization of the property of having a dominated splitting for switched linear systems. This characterization can be represented as the contraction of a set of "generalized cones" by the system; see Figure 6 for an illustration. Note that similar (and less similar) geometric approaches are used for instance to study **positive** or **differentially positive** systems (see, e.g., Grussler and Rantzer, 2014, Forni and Sepulchre, 2016, and Forni et al., 2017), or systems whose asymptotic behavior is low-dimensional, called ***p*-dominant systems** (see Forni and Sepulchre, 2019). By extending the property of p-dominant systems to switched linear systems and combining it with **graph-theoretic tools** to increase its expressiveness, we provide an asymptotically non-conservative algorithmic framework for the study of the property of having a dominated splitting for switched linear systems; in particular, in this framework, the generalized cones are computed using convex optimization (a field of applied mathematics that has proved useful in many areas of systems and control theory) and the relations of contraction are enforced by a graph capturing the possible sequences of modes of the system.

We also study the property of dominance for <u>nonlinear systems</u>. The goal is to study those systems whose "linearized" dynamics can be split into a dom-

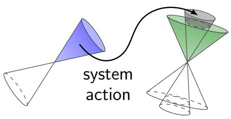


Figure 6: Two generalized cones (blue and green). The blue cone is contracted into the green one by the action of one of the modes of the system.

inant component and a dominated component. For that, we use the theory of dominant switched linear systems, introduced before. In particular, by leveraging the algorithmic approach for dominance analysis of switched linear systems and combining it with techniques from **abstraction** (a well-known tool in control theory, allowing for instance to abstract a nonlinear system as a collection of "locally defined" linear systems), we provide an algorithmic framework for the verification of the property of dominance for nonlinear systems.

Finally, we describe several applications of the property of dominance for switched linear systems and nonlinear systems; for instance,

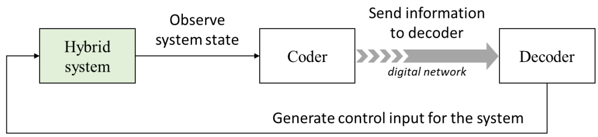
- for the study of the convergence of the trajectories to a low-dimensional time-varying attractor (with applications for instance in population dynamics);
- for the robustness analysis of attractors of nonlinear systems (via the notion of hyperbolicity, which is a special case of dominance for nonlinear systems);
- for the computation of data rate requirements for the observation and control of switched linear systems and nonlinear systems (this application is related with the second main topic of this thesis, namely the "quantized control of hybrid systems", introduced below).

Quantized control of hybrid systems

We study the impact of quantization and limited information for the observation and control of hybrid systems. Although hybrid systems and quantized control have been two active research areas for some time now, the combination of these two aspects in control problems has received limited attention so far; see, e.g., Nair et al. (2003), Xiao et al. (2010), Liberzon (2014), Vicinansa and Liberzon (2019) and Yang et al. (2020). The study of these two aspects in a unified framework is however essential to tackle numerous modern control problems, encountered for instance in cyber-physical systems.

In this thesis, we focus on the quantized control of switched linear systems (a paradigmatic class of hybrid systems). To do that, we study the existence of "devices", called **coders–decoders**, that stabilize or estimate the state of the system by exchanging information at a limited data rate; see Figure 7 for an illustration. In particular, we are interested in deriving bounds on the communication data rate that is needed between the coder and the decoder to ensure observability or stabilizability of the system.

Figure 7: Control of a hybrid system with a coder–decoder.



The main difficulty arising from the presence of a switching behavior in quantized control problems is that the coder–decoder does not know in advance when the switching will occur, and thus, since the effect of the switching is abrupt, the coder cannot code the future state of the system and send this information to the decoder. This effect can be mitigated if the decoder **knows in real time when a switching occurs** (which can be achieved for instance if one uses an event-triggered communication protocol, or if the decoder can choose the mode of the system); or if the switching signal satisfies some conditions (typically, *slow-switching conditions*) ensuring its **trackability**.

In this thesis, we study the observability and controllability of switched linear systems under data-rate constraints, in both of these settings. Namely,

- for the first setting (the decoder knows the current mode in real time), we study the relation between the *topological entropy* (a well-known concept in systems theory) and the minimal data rate for state estimation and stabilization of switched linear systems. In particular, we introduce the notion of **worst-case topological entropy** for switched linear systems. This quantity can be expressed as the the maximal growth rate of the system when seen as acting on elements of volumes (formalized using the notion of exterior algebra, introduced above); this allows us to provide an algorithmic framework for the theoretical characterization and practical computation of optimal quantizing–controlling strategies for switched linear systems.

- for the second setting (the decoder does not know the current mode in real time), we provide several new results regarding the observability and stabilizability of switched linear systems under data-rate constraints and the importance of **switching signal's trackability** for this problem. In particular, we show that these systems are in general not stabilizable by a coder–decoder with finite communication data rate, if the switching signal can switch arbitrarily fast. On the other hand, we show that, under mild slow-switching conditions, the system can be stabilized with a finite data rate and we describe the implementation of a coder–decoder achieving stabilization as well as an upper bound on the required data rate as a function of the slow-switching parameters. We also present results regarding the observability of these systems under data-rate constraints; namely, we show that these systems are in general not observable by a coder–decoder with finite communication data rate, even if the switching signal is constrained to switch arbitrarily slow.

Outline of the thesis

The thesis is organized as follows. See also Figure 8 for a representation of the dependencies between the different sections of this thesis.

In Chapter 1, we introduce the necessary background. In Section 1.1, we introduce the important concepts and results related to hybrid systems; this class of systems provides a general and flexible framework for the study of a wide range of phenomenons and concrete systems, including switched linear systems, smooth dynamical systems and networked systems. Then, in Section 1.2, we remind several important concepts and results from the theory of smooth dynamical systems. In Section 1.3, we do the same with the theory of switched systems. In Section 1.4, we remind some basic notions from the theory of abstraction (aka. symbolic control) of dynamical systems, which will be useful for the study of dominant smooth dynamical systems. Finally, in Section 1.5, we introduce several concepts and results related to the networked control of hybrid systems.

In Chapter 2, we present the first part of our contributions, which deals with the dominance analysis of switched linear systems and smooth nonlinear systems. In Section 2.1, we introduce the topic and review the literature. In Section 2.2, we present our contributions for the study of dominant switched linear systems. In Section 2.3, we do the same for dominant nonlinear systems. Finally, in Section 2.4, we present our conclusions and perspectives for future research on this topic.

In Chapter 3, we present the second part of our contributions, which deals with the quantized control of switched linear systems. In Section 3.1, we introduce the topic and review the literature. In Section 3.2, we present our contributions for the study of the quantized control of linear time-varying systems. In Section 3.3, we present our contributions for the study of the quantized control of switched linear systems in the first setting (called the “mode-dependent” setting). In Section 3.4, we do the same for the second setting (called the “mode-oblivious” setting). Finally, in Section 3.5, we present our conclusions and perspectives for future research on this topic.

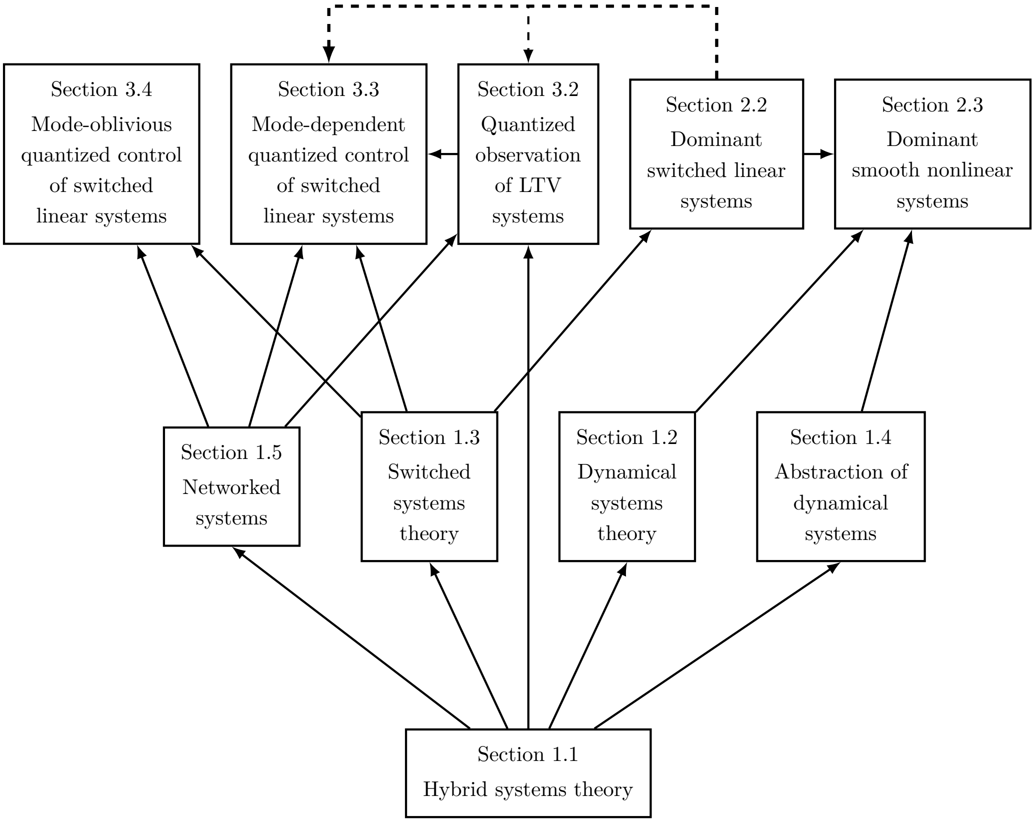


Figure 8: Dependencies between the different sections of the thesis. Dashed arrows represent soft dependencies, in the sense that the section at the origin of the arrow is needed only for some subsections of the section at the end of the arrow.

List of publications

Conference publications

Guillaume O Berger, Fulvio Forni, and Raphaël M Jungers. Path-complete p-dominant switching linear systems. In *2018 IEEE 57th IEEE Conference on Decision and Control (CDC)*, pages 6446–6451. IEEE, 2018. doi: 10.1109/CDC.2018.8619703.

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Guillaume O Berger, Raphaël M Jungers, and Zheming Wang. Chance-constrained quasi-convex optimization with application to data-driven switched systems control. In *Proceedings of the 3rd Conference on Learning for Dynamics and Control*, volume 144 of *Proceedings of Machine Learning Research*, pages 571–583. PMLR, 2021. `http://proceedings.mlr.press/v144/berger21a.html`.

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Guillaume O Berger and Raphaël M Jungers. Formal methods for computing hyperbolic invariant sets for nonlinear systems. *IEEE Control Systems Letters*, 4(1):235–240, 2020c. doi: 10.1109/LCSYS.2019.2923923.

Guillaume O Berger, P-A Absil, Raphaël M Jungers, and Yurii Nesterov. On the quality of first-order approximation of functions with Hölder continuous gradient. *Journal of Optimization Theory and Applications*, 185:17–33, 2020. doi: 10.1007/s10957-020-01632-x.

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List of symbols

Below are some generic symbols and terminology used throughout the thesis. Notation specific to the different chapters of the thesis will be introduced in due course of the text.

For real numbers, "positive", "negative", "nonnegative" and "nonpositive" mean "strictly larger than zero", "strictly smaller than zero", "larger than or equal to zero" and "smaller than or equal to zero", respectively.

Sets

$\emptyset$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$	The empty set, the sets of integers, nonnegative integers, real numbers and complex numbers
$\{x \in A : \mathcal{P}(x)\}$	Set defined by comprehension: contains all elements of A satisfying $\mathcal{P}$
2^A	The power set of A

Functions

$f : A \to B$	A function from A to B
B^A	The set of functions from A to B
$\min A$, $\min_{x\in A} f(x)$	The minimum of A and the minimum of f over A
$\max A$, $\max_{x\in A} f(x)$	The maximum of A and the maximum of f over A
$\inf A$, $\inf_{x\in A} f(x)$	The infimum of A and the infimum of f over A
$\sup A$, $\sup_{x\in A} f(x)$	The supremum of A and the supremum of f over A
$\arg\min_{x\in A} f(x)$	The minimizer of f over A
$\arg\max_{x\in A} f(x)$	The maximizer of f over A
$\lim_{x\to a} f(x)$	The limit of f at a
$\lim_{x\to a^-} f(x)$	The left limit of f at a

$\lim_{x \to a^+} f(x)$	The right limit of f at a
$\liminf_{x \to a} f(x)$	The limit inferior of f at a
$\limsup_{x \to a} f(x)$	The limit superior of f at a

Norms

$\lvert a \rvert$, $\lvert A \rvert$	The absolute value (aka. modulus) of $a \in \mathbb{C}$ and the cardinality of the set A
$\lVert x \rVert$, $\lVert M \rVert$	The Euclidean norm of $x \in \mathbb{R}^n$ and the spectral norm of $M \in \mathbb{R}^{m \times n}$

Others

$\mathrm{Re}(a)$, $\mathrm{Im}(a)$	The real part and the imaginary part of $a \in \mathbb{C}$
$x^{(i)}$	The i^{th} component of the vector x

Chapter 1

Preliminaries

In this chapter, we introduce the necessary background for the presentation of the contributions of this thesis, described in the subsequent chapters. This chapter needs not be read completely before the other chapters, but can be referred to in due time.

1.1 Hybrid systems

Hybrid systems are dynamical systems resulting from the interaction of continuous and discrete dynamics. The setting of hybrid systems provides a very general and flexible framework to study a wide range of phenomenons and concrete systems. For instance, smooth dynamical systems and switched linear systems, which will be studied into details in this thesis, are both special instances of hybrid systems. Similarly, devices such as coders–decoders, which will be instrumental in our study of networked control systems, are essentially hybrid systems. Having a single framework to study different classes of systems allows to reduce the redundancy in the definitions and results, and more importantly to better see the connections between the different concepts and results, and to identify possible generalizations to other classes of systems. In particular, we define the notions of trajectories, stability, convergence and control for the general framework of hybrid systems, as these notions will be instrumental in many different contexts throughout this thesis.

The section is organized as follows. In Subsection 1.1.1, we introduce the concepts of hybrid system and trajectories of these systems. In Subsection 1.1.2, we introduce the classes of smooth dynamical systems, switched systems and time-varying dynamical systems. Finally, in Subsection 1.1.3, we discuss the notions of stability and control of hybrid systems.

References. The literature on hybrid systems is quite vast; see, e.g., van der Schaft and Schumacher (2000), Goebel et al. (2012), Alur (2015) and Lee and Seshia (2017) for introductions. Our principal reference for the definition of hybrid systems and their trajectories is Goebel et al. (2012). However, we adapt the definition of the trajectories of hybrid systems to consider systems with input and to have a *single* time variable (vs. a "hybrid time domain" as in Goebel et al., 2012); the latter will be important for the study of these systems under communication constraints (see Section 1.5).

Notation. A set-valued map F from A to B is denoted by $F : A \rightrightarrows B$. The domain of a set-valued map $F : A \rightrightarrows B$, denoted by $\operatorname{dom} F$, is defined as $\operatorname{dom} F = \{x \in A : F(x) \neq \emptyset\}$. The restriction of a function $f : A \to B$ to a set $A' \subseteq A$ is denoted by $f|_{A'}$.

1.1.1 Hybrid systems and their trajectories

We start with the concepts of hybrid system and trajectories of these systems.

Definition 1.1 (Hybrid system)**.** *A* hybrid system *is a quintuple* (X, X_0, U, F, G) *where*

- $X \subseteq \mathbb{R}^n$ *is a nonempty set called the* state space *of the system;*
- $X_0 \subseteq X$ *is a nonempty set called the* initial set*;*
- U *is a nonempty set called the* input space *of the system;*
- $F : X \times U \rightrightarrows \mathbb{R}^n$ *is a set-valued function called the* flow map *of the system;*
- $G : X \times U \rightrightarrows X$ *is a set-valued function called the* jump map *of the system.*

If the initial set X_0 *is the whole state space* X*, then we will denote the hybrid system simply by the quadruple* (X, U, F, G)*.*

The above definition deserves the following explanations (which will also become clearer with the definition of a trajectory of a hybrid system; see Definition 1.3 below). The state space X is the set in which live the variables of the system, meaning that the variables describing the state of the system (e.g., the position, the velocity, the gear ratio, etc., in the case of a self-driving car) belong to the set X. The initial set X_0 contains the possible values of the variables of the system at the initial time (i.e., at $t = 0$). The external inputs (e.g., the accelerator intensity or the wind force in the case of a self-driving car) are described as functions of time taking values in the set U. The flow map F describes how the derivative of the variables behaves when they evolve "smoothly" (e.g., in the case of a self-driving car, the derivative of the position is equal to the velocity, and the derivative of the velocity depends on the accelerator intensity and the wind force). At some discrete times, the variables of the system can also evolve "abruptly" (aka. "jump"). Such abrupt transitions of the variables are described by the jump map G.

The flow map and the jump map of a hybrid system are set-valued functions. This allows for great expressiveness of these systems by enabling that the derivative and the jumps of the trajectories (see Definition 1.3 below) can take different values at a same point $(x, u) \in X \times U$; this property, called *non-determinism*, is instrumental for instance for systems with unknown disturbances. It also allows to restrict the flow set and the jump set of trajectories (see Definition 1.3 below) by letting F and G be nonempty only on subsets of $X \times U$. See also Example 1.1 for an illustration.

A hybrid system (X, X_0, U, F, G) for which U is a singleton cannot be influenced by external inputs. Such systems will be said to be *autonomous*. On the other hand, if we want to emphasize that a given hybrid system is affected by external inputs, we will say that the system is *non-autonomous*.

We define below the concept of trajectories of a hybrid system. But, first of all, it is important to specify which kinds of input functions are acceptable for the system. In this thesis, we restrict our attention to piecewise continuous input functions.

Definition 1.2 (Piecewise continuous input functions). *Consider a set $U \subseteq \mathbb{R}^m$ and an input function $u : \mathbb{R}_{\geq 0} \to U$. We say that u is* piecewise continuous *if for any $T \in \mathbb{R}_{\geq 0}$, there is a finite sequence of times $(\tau_j)_{j=0}^{J+1} \subseteq \mathbb{R}$, $0 = \tau_0 < \tau_1 < \ldots < \tau_{J+1} = T$, such that for every $j \in \{0, \ldots, J\}$, u is continuous on (τ_j, τ_{j+1}) and has a right limit in τ_j and a left limit in τ_{j+1}.*

Given a hybrid system $\mathsf{HySys} = (X, X_0, U, F, G)$, we denote by $\mathcal{U}(\mathsf{HySys})$ (or simply $\mathcal{U}$ if HySys is clear from the context) the set of piecewise continuous functions from $\mathbb{R}_{\geq 0}$ to U. This space of input functions is at the same time general enough to capture a large range of behaviors (any integrable function can be approximated arbitrarily well by a piecewise continuous function in the L^1 topology) and sufficiently regular to guarantee (at least local) existence of trajectories of hybrid systems.

Definition 1.3 (Trajectory of a hybrid system). *Given a hybrid system $\mathsf{HySys} = (X, X_0, U, F, G)$ and an input function $u \in \mathcal{U}$, a* trajectory *of HySys with input u is a function $\phi : E \to X$, with domain $E \subseteq \mathbb{R}$, satisfying that for any $T \in E$, (i) $(-\infty, T] \subseteq E$, and (ii) there is a finite sequence of times $(\tau_j)_{j=0}^{J+1} \subseteq \mathbb{R}$,*

$0 = \tau_0 < \tau_1 < \ldots < \tau_{J+1} = T$, *such that*

- *for every* $j \in \{0, \ldots, J\}$, ϕ *is absolutely continuous on* $[\tau_j, \tau_{j+1})$ *and satisfies*

$$\begin{cases} (\phi(t), u(t)) \in \operatorname{dom} F & \text{for all } t \in [\tau_j, \tau_{j+1}), \\ \dot{\phi}(t) \doteq \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \in F(\phi(t), u(t)) & \text{for almost all } t \in [\tau_j, \tau_{j+1}); \end{cases} \tag{1.1}$$

- *for every* $j \in \{0, \ldots, J+1\}$, ϕ *has a left limit at* τ_j, *and satisfies*

$$\phi(\tau_j) \in G(\phi(\tau_j-), u(\tau_j)), \qquad \text{where } \phi(\tau_j-) = \lim_{t \to \tau_j^-} \phi(t), \tag{1.2}$$

  or satisfies $\phi(\tau_j) \in G(\phi(\tau_j-), u(\tau_j)) \cup \{\phi(\tau_j-)\}$ *if* $j \in \{0, J+1\}$;

and $\phi(0-) \in X_0$.

In other words, the domain E of a trajectory $\phi : E \to X$ of a hybrid system (X, X_0, U, F, G) can be decomposed into a finite or an infinite union of disjoint intervals: $E = (-\infty, 0) \cup [0, \tau_1) \cup \cdots \cup [\tau_j, \tau_{j+1}) \cup \cdots \cup [\tau_N, \tau_{N+1})$ or $E = (-\infty, 0) \cup [0, \tau_1) \cup \cdots \cup [\tau_j, \tau_{j+1}) \cup \cdots \cup [\tau_N, \tau_{N+1}) \cup \{\tau_{N+1}\}$ or $E = (-\infty, 0) \cup [0, \tau_1) \cup \cdots \cup [\tau_j, \tau_{j+1}) \cup \cdots$. The intervals $[\tau_j, \tau_{j+1})$ are called the *flow periods* of ϕ, and ϕ satisfies (1.1) on these intervals. The events τ_j at which ϕ satisfies (1.2) are called the *jumps* (or *switches*) of ϕ. Note that the part of ϕ defined on $(-\infty, 0)$ is relevant *only* for the value of $\phi(0-)$; we will say that $\phi(0-)$ is the *initial condition* of ϕ, or that ϕ *starts at* $\phi(0-)$. See also Figure 1.1 for an illustration.

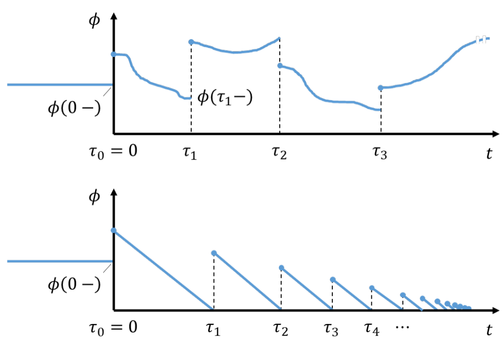

Figure 1.1: Examples of trajectories of a hybrid system. The first example shows a non-Zeno trajectory, while the second one shows a Zeno trajectory (see Definition 1.4).

We distinguish different types of trajectories of a hybrid system, depending on whether they can be *extended* to "longer-term" trajectories, or depending on their domain and the occurrences of flow and jump periods.

Definition 1.4 (Types of trajectories). *Consider a hybrid system* HySys $=$ (X, X_0, U, F, G) *and a trajectory* $\phi : E \to X$ *of* HySys *with input* $u \in \mathcal{U}$. ϕ *is* *said to be* maximal *if there is no trajectory* $\phi' : E' \to X$ *of* HySys *with input* u*, such that* $E \subsetneq E'$ *and* $\phi = \phi'|_E$*.* ϕ *is said to be* complete *if* $E = \mathbb{R}$*.* ϕ *is said to be* non-Zeno *if it is complete or if there is* $T \in E$ *such that* $[T, \infty) \cap E$ *is a flow period of* ϕ*; otherwise* ϕ *is said to be* Zeno.

In the following of this thesis, we restrict our attention to *maximal* trajectories of hybrid systems, and we refer to them simply as *trajectories*.

Examples of hybrid systems include complex systems, such as walking robots or cyber-physical systems, but also simple mechanical systems such as bouncing balls. We present below the academic example of the bouncing ball (see, e.g., Goebel et al., 2012, Example 1.1) to illustrate the concepts of hybrid systems and their trajectories.

Example 1.1 (Bouncing ball). Consider a ball (modeled as a point-mass m) bouncing on the floor; see Figure 1.2 for an illustration. When the ball hits the floor, it bounces upwards with a velocity that is slightly smaller than the velocity it had just before hitting the floor, due to energy dissipation: namely, after the bounce, the velocity is decreased by a factor $\gamma \in (0, 1)$. Between two bounces, the ball is subject only to the gravity (attracting it downwards with acceleration g) and to an ascending wind with "force" $u \geq 0$ (thereby pushing the ball upwards with acceleration u/m). The force of the wind is considered as an input of the system, so that the input space U is equal to $\mathbb{R}_{\geq 0}$.

The dynamics of the ball can be modeled as a hybrid system as follows. The variables describing the movement of the ball are its height x and its vertical velocity v (i.e., v is the time derivative of x). When the ball touches the floor and its velocity is downwards (that is, when $x = 0$ and $v < 0$), then the ball bounces, so that its velocity becomes $-\gamma v$. Hence, the jump map (describing the bouncing event) is defined by $G(x, v, u) = \{(0, -\gamma v)\}$ for all $x = 0$, $v < 0$ and $u \geq 0$, and $G(x, v, u) = \emptyset$ for all other values of x, v and u. When the ball is in the air, or touches the floor and goes upwards (that is, when $x > 0$, or when $x = 0$ and $v > 0$), then the ball is subject to the gravity and the wind force, so that its acceleration (that is, the time derivative of v) increases with a rate equal to $u/m - g$. Hence, the flow map (describing the free fall of the ball) is defined by $F(x, v, u) = \{(v, u/m - g)\}$ for all $x > 0$, v and $u \geq 0$ and for all $x = 0$, $v > 0$ and $u \geq 0$, and $F(x, v, u) = \emptyset$ for all other values of x, v and u.

1.1.2 Special classes of hybrid systems

In this subsection, we discuss some specific classes of hybrid systems that are relevant for this work.

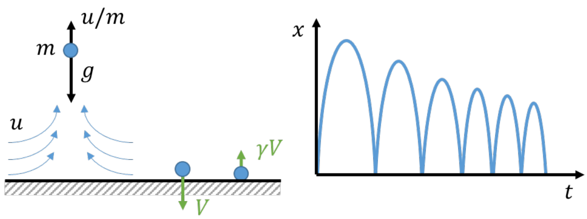

Figure 1.2: Bouncing ball. *Left:* The black arrows are the forces acting on the ball when it is in the air. The green arrows represent the velocity of the ball before and after the bounce. *Right:* Height of the ball as a function of time. Any trajectory of the ball is a solution to the hybrid system described in Example 1.1.

Smooth dynamical systems

The first class of hybrid systems that we will consider is the one of smooth dynamical systems. Let us start with *continuous-time smooth dynamical systems.* These systems are described by *differential equations* of the form

$$\dot{\xi}(t) \doteq \frac{\mathrm{d}}{\mathrm{d}t}\xi(t) = f(\xi(t)) \qquad \text{for all } t \in \mathbb{R}_{\geq 0},$$

where $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is uniformly Lipschitz continuous and infinitely differentiable[1]. As for the discrete time: *discrete-time smooth dynamical systems* are described by *difference equations* of the form

$$\xi^+(t) \doteq \xi(t+1) = f(\xi(t)) \qquad \text{for all } t \in \mathbb{N},$$

where $\xi : \mathbb{N} \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is infinitely differentiable[2].

Smooth dynamical systems are encountered in many natural processes, such as biology, physics, opinion dynamics, etc. We give below a simple example from population dynamics to illustrate the concept of discrete-time smooth dynamical system.

Example 1.2 (Population dynamics). Consider a population evolving in an environment with limited food supply. The *transmission* rate (i.e., *survival* rate + *fertility* rate) decreases proportionally with the population (because food is limited, so that when the population is too large, the individuals starve and are less fertile); more precisely, the transmission rate is equal to $r_1 - r_2\xi(t)$ where

[1]For most applications, assuming that f is only continuously differentiable would be enough. However, the definition with f infinitely differentiable has the advantage of defining a class of systems that is closed under linearization (see Subsection 1.2.3). Hence, for the sake of simplicity, we restrict our attention to this class of systems, since the assumption that f is infinitely differentiable is not limiting for the applications considered in this thesis.

[2]Same comment as for the continuous time.

$\xi(t)$ is the number of individuals at time t, and $r_1 > 0$ and $r_2 > 0$ are parameters. The evolution of the population can thus be described by the following discrete-time smooth dynamical system: $\xi(t+1) = (r_1 - r_2\xi(t))\xi(t)$; see Figure 1.3 for an illustration. This system is often referred to as the *logistic map* and is known to exhibit a large range of behaviors, from multi-stability to chaotic behavior (see, e.g., Strogatz, 2015).

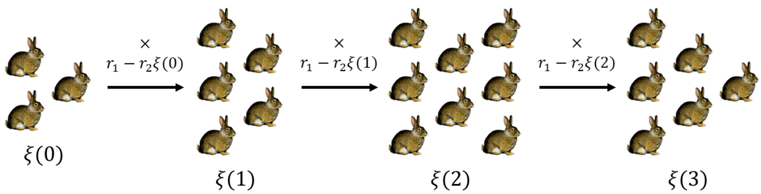

Figure 1.3: Evolution of rabbit population in an environment with limited food supply. The time instants are discrete ($t = 0, 1, 2, \dots$) and correspond to months. The population (number of individuals) $\xi(t)$ evolves according to the logistic map: $\xi(t+1) = (r_1 - r_2\xi(t))\xi(t)$.

Continuous-time and discrete-time smooth dynamical systems can be defined using the formalism of hybrid systems as follows.

Definition 1.5 (Continuous-time smooth dynamical system)**.** *A* continuous-time smooth dynamical system *is an autonomous hybrid system* $\mathsf{HySys} = (\mathbb{R}^n, \{0\}, F, G)$ *where* $F(x, 0) = \{f(x)\}$ *for all* $x \in \mathbb{R}^n$ *with* $f : \mathbb{R}^n \to \mathbb{R}^n$ *uniformly Lipschitz continuous and infinitely differentiable, and* $G : \mathbb{R}^n \times \{0\} \rightrightarrows \emptyset$ *is the empty map.*

Definition 1.6 (Discrete-time smooth dynamical system)**.** *A* discrete-time smooth dynamical system *is an autonomous hybrid system* $(\hat{X}, \hat{X}_0, \{0\}, F, G)$ *where* $\hat{X} = \mathbb{R}^n \times [0, 1]$*,* $\hat{X}_0 = \mathbb{R}^n \times \{0\}$*,* $F(x, \tau, 0) = \{0\} \times \{1\}$ *for all* $(x, \tau) \in \mathbb{R}^n \times [0, 1]$*, and* $G(x, 1, 0) = \{f(x)\} \times \{0\}$ *for all* $x \in \mathbb{R}^n$ *with* $f : \mathbb{R}^n \to \mathbb{R}^n$ *infinitely differentiable and* $G(x, \tau, 0) = \emptyset$ *for all* $(x, \tau) \in \mathbb{R}^n \times [0, 1)$*.*

Remark 1.1. In Definition 1.6, the variable "τ" measures the time between two *jumps* since its time derivative is equal to one during flow periods (see the definition of F) and it is mapped to zero at each jump (see the definition of G). Similarly, the definition of F implies that the variable "x" is constant between two jumps. Finally, the requirement that $G(x, \tau, 0) = \emptyset$ when $\tau < 1$ implies that jumps occur with an interval of exactly one unit of time between them, so that the variable "x" can change only at integer times, hence the name *discrete-time* systems.

Continuous-time and discrete-time smooth dynamical systems are more concisely described by specifying only the state space $\mathbb{R}^n$ and the map f. Therefore, in the following, we will denote these systems simply by the ordered pair $(\mathbb{R}^n, f)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is infinitely differentiable (and uniformly Lipschitz continuous if the system is continuous-time).

Switched systems

The second class of hybrid systems in which we will be interested is the one of switched systems. Let us start with *continuous-time switched systems*. These are systems of the form

$$\dot{\xi}(t) = f_{\sigma(t)}(\xi(t), u(t)) \qquad \text{for almost all } t \in \mathbb{R}_{\geq 0},$$

where $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $\sigma : \mathbb{R}_{\geq 0} \to \Sigma \doteq \{1, \ldots, N\}$ and for each $i \in \Sigma$, $f_i : \mathbb{R}^n \times U \to \mathbb{R}^n$ is uniformly Lipschitz continuous in its first argument. As for the discrete time: *discrete-time switched systems* are systems of the form

$$\xi^+(t) \doteq \xi(t+1) = f_{\sigma(t)}(\xi(t), u(t)) \qquad \text{for all } t \in \mathbb{N},$$

where $\xi : \mathbb{N} \to \mathbb{R}^n$, $\sigma : \mathbb{N} \to \Sigma \doteq \{1, \ldots, N\}$ and for each $i \in \Sigma$, $f_i : \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuous in its first argument. The set Σ is called the set of *modes* of the system. The trajectories of a switched system are given by the ordered pair (ξ, σ) where $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is absolutely continuous and $\sigma : \mathbb{R}_{\geq 0} \to \Sigma$ is right-continuous and piecewise constant[3] (in continuous time) or $\xi : \mathbb{N} \to \mathbb{R}^n$ and $\sigma : \mathbb{N} \to \Sigma$ (in discrete time); ξ is called the *continuous variable* and σ is called the *switching signal*.

The description of a switched system also often involves conditions that must be satisfied by the switching signal; see, e.g., Example 1.3 below and Subsection 1.3.1 for examples of such conditions.

Example 1.3 (Addition with packet dropouts). Consider a simple discrete-time system consisting in adding numbers generated sequentially by a user: that is, the input is a function $u : \mathbb{N} \to \mathbb{R}$ (i.e., the sequence of numbers $u(0), u(1), \ldots$) and the goal of the system is to compute at each time $t \in \mathbb{N}$ the sum $\xi(t) = \sum_{j=0}^{t-1} u(j)$ (with the convention that $\xi(0) = 0$). This can be implemented by the following discrete-time system: $\xi(t+1) = \xi(t) + u(t)$.

Now, assume that the numbers $u(t)$ are sent to the system over a noisy channel, so that it may happen that a number $u(t)$ never arrives to the system

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[3] This (being right-continuous and piecewise constant) means that for any $T \in \mathbb{R}_{\geq 0}$, there is a finite sequence of times $(\tau_j)_{j=0}^{J+1} \subseteq \mathbb{R}$, $0 = \tau_0 < \tau_1 < \ldots < \tau_{J+1} = T$, such that for every $j \in \{0, \ldots, J\}$, σ is constant on $[\tau_j, \tau_{j+1})$.

due to packet dropouts; see Figure 1.4 for an illustration. In that case, the system will not add the number (this is equivalent to assuming that $u(t) = 0$). This system can be described as a discrete-time switched system with two modes:

- Mode 1 describing the event when there is no dropout; in that case, $\xi(t+1) = \xi(t) + u(t)$;
- Mode 2 describing the event when there is a dropout; in that case, $\xi(t+1) = \xi(t)$.

Finally, in some cases, code-correction algorithms can be used to ensure that packet dropouts are sufficiently spaced in time; for instance we may assume that between two packet dropouts, there are at least m correctly sent packets. These types of assumptions can be included in the description of the system as well: namely, by imposing conditions on the switching signal of the system (in our case, we impose that between any two time instants at which the switching signal is equal to 1, there are at least m instants at which it is equal to 2).

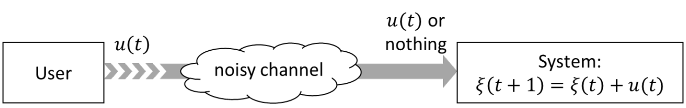

Figure 1.4: Addition of numbers over a noisy channel.

Continuous-time and discrete-time switched systems can be defined using the formalism of hybrid systems as follows.

Definition 1.7 (Continuous-time switched system)**.** *A* continuous-time switched system *is a hybrid system* $(\hat{X}, U, F, G)$ *where* $\hat{X} = \mathbb{R}^n \times \Sigma \times Y$ *with* $\Sigma = \{1, \dots, N\}$ *and* $Y \subseteq \mathbb{R}^\ell$*,* $U \subseteq \mathbb{R}^m$*,* $F(x, i, y, u) = \{f(x, i, u)\} \times \bar{F}(i, y, 0)$ *for all* $(x, i, y, u) \in \mathbb{R}^n \times \Sigma \times Y \times U$ *with* $f : \mathbb{R}^n \times \Sigma \times U \to \mathbb{R}^n$ *uniformly Lipschitz continuous in its first argument and* $\bar{F} : \Sigma \times Y \times \{0\} \rightrightarrows \{0\} \times \mathbb{R}^\ell$*, and* $G(x, i, y, u) = \{x\} \times \bar{G}(i, y, 0)$ *for all* $(x, i, y, u) \in \mathbb{R}^n \times \Sigma \times Y \times U$ *with* $\bar{G} : \Sigma \times Y \times \{0\} \rightrightarrows \Sigma \times Y$*.*

Definition 1.8 (Discrete-time switched system)**.** *A* discrete-time switched system *is a hybrid system* $(\hat{X}, \hat{X}_0, U, F, G)$ *where* $\hat{X} = \mathbb{R}^n \times [0,1] \times \Sigma \times Y$ *with* $\Sigma = \{1, \dots, N\}$ *and* $Y \subseteq \mathbb{R}^\ell$*,* $\hat{X} = \mathbb{R}^n \times \{0\} \times \Sigma \times Y$*,* $U \subseteq \mathbb{R}^m$*,* $F(x, \tau, i, y, u) = \{0\} \times \{1\} \times \{0\} \times \{0\}$ *for all* $(x, \tau, i, y, u) \in \mathbb{R}^n \times [0,1] \times \Sigma \times Y \times U$*,* $G(x, 1, i, y, u) = \{f(x, i, u)\} \times \{0\} \times \bar{G}(i, y, 0)$ *for all* $(x, i, y, u) \in \mathbb{R}^n \times \Sigma \times Y \times U$ *with* $f : \mathbb{R}^n \times \Sigma \times U \to \mathbb{R}^n$ *continuous in its first argument and* $\bar{G} : \Sigma \times Y \times \{0\} \rightrightarrows \Sigma \times Y$*, and* $G(x, \tau, i, y, u) = \emptyset$ *for all* $(x, \tau, i, y, u) \in \mathbb{R}^n \times [0,1) \times \Sigma \times Y \times U$*.*

Remark 1.2. See Remark 1.1 for the motivation of the name *discrete-time* switched system.

The above definitions deserve the following explanations. Definition 1.7 tells us that a continuous-time switched system can be decomposed in two parts: (i) an autonomous hybrid system $(\Sigma \times Y, \{0\}, \bar{F}, \bar{G})$ describing the evolution of the second component of the trajectories (called the switching signal), and (ii) a continuous-time non-autonomous dynamical system $(\mathbb{R}^n, U \times \Sigma, f)$ controlled by the input and the switching signal. In particular, it is via the subsystem (i) that constraints are imposed on the switching signal in the definition of a continuous-time switched system. A similar decomposition holds for discrete-time switched system.

Continuous-time and discrete-time switched systems can be more concisely described by specifying only the state space $\mathbb{R}^n$, the input space U and functions f_i, for each $i \in \Sigma$, plus a description of the set of admissible switching signals of the system. Therefore, in the following, we will denote these systems simply by the triple $(\mathbb{R}^n, U, \{f_i\}_{i\in\Sigma})$ where for each $i \in \Sigma$, $f_i : \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuous (or uniformly Lipschitz continuous if the system is continuous-time) in its first argument, plus a description of the set of admissible switching signals of the system.

The description of the admissible switching signals can take different forms and we will discuss some of them in Subsection 1.3.1.

Remark 1.3. Switched systems can also be defined as special instances of *hybrid automata*, which are essentially graphs with continuous dynamics (described by flow maps) associated to the vertices and with *pre* and *post* conditions (often called *guards* and *reset maps*) associated to the edges; see, e.g., Mitra (2021). In this thesis, we defined switched systems using the template of hybrid systems; the goal being to avoid multiplying the concepts and the notation, by sticking to a single template in which we define different classes of systems and other concepts useful for this work.

Time-varying dynamical systems

The last class of hybrid systems in which we will be interested is the one of time-varying dynamical systems. Let us start with *continuous-time time-varying dynamical systems.* These systems are described by *time-dependent differential equations* the form

$$\dot{\xi}(t) \doteq \frac{\mathrm{d}}{\mathrm{d}t}\xi(t) = f(t, \xi(t)) \qquad \text{for all } t \in \mathbb{R}_{\geq 0},$$

where $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in its first argument and uniformly Lipschitz continuous in its second argument. As for the discrete time: *discrete-time time-varying dynamical systems* are described by *time-dependent difference equations* of the form

$$\xi^+(t) \doteq \xi(t+1) = f(t, \xi(t)) \qquad \text{for all } t \in \mathbb{N},$$

where $\xi : \mathbb{N} \to \mathbb{R}^n$ and $f : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in its second argument.

Continuous-time and discrete-time time-varying dynamical systems can be defined using the formalism of hybrid systems as follows.

Definition 1.9 (Continuous-time time-varying dynamical system). *A* continuous-time time-varying dynamical system *is an autonomous hybrid system* HySys $= (\mathbb{R}^{n+1}, \hat{X}_0, \{0\}, F, G)$ *where* $\hat{X}_0 = \mathbb{R}^n \times \{0\}$, $F(x, t, 0) = \{f(t, x)\} \times \{1\}$ *for all* $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ *with* $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *continuous in its first argument and uniformly Lipschitz continuous in its second argument, and* $G : \mathbb{R}^n \times \mathbb{R} \times \{0\} \rightrightarrows \emptyset$ *is the empty map.*

Definition 1.10 (Discrete-time time-varying dynamical system). *A* discrete-time time-varying dynamical system *is an autonomous hybrid system* $(\hat{X}, \hat{X}_0, \{0\}, F, G)$ *where* $\hat{X} = \mathbb{R}^{n+1} \times [0, 1]$, $\hat{X}_0 = \mathbb{R}^n \times \{0\} \times \{0\}$, $F(x, t, \tau, 0) = \{0\} \times \{0\} \times \{1\}$ *for all* $(x, t, \tau) \in \mathbb{R}^n \times \mathbb{R} \times [0, 1]$*, and* $G(x, t, 1, 0) = \{f(t, x)\} \times \{t+1\} \times \{0\}$ *for all* $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ *with* $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *continuous in its second argument and* $G(x, t, \tau, 0) = \emptyset$ *for all* $(x, t, \tau) \in \mathbb{R}^n \times \mathbb{R} \times [0, 1)$.

Remark 1.4. See Remark 1.1 for the motivation of the name *discrete-time* time-varying system.

Continuous-time and discrete-time time-varying dynamical systems are more concisely described by specifying only the state space $\mathbb{R}^n$ and the map f. Therefore, in the following, we will denote these systems simply by the ordered pair $(\mathbb{R}^n, f)$ where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in its first argument and uniformly Lipschitz continuous in its second argument (if the system is continuous-time), or where $f : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in its second argument (if the system is discrete-time).

1.1.3 Stability and control of hybrid systems

Stability theory deals with the study of the convergence properties of the trajectories of hybrid systems. Several notions of stability coexist, accounting for different types of convergence; e.g., *internal stability* (convergence of the trajectories to a point in the state space), *set stability* (convergence of the trajectories to a set in the state space), and *output stability* (convergence of the trajectories after transformation by a function called the *output map*).

In particular, the notion of output stability is often instrumental for the study of hybrid systems, as many applications involving these systems require that only a subset of the state variables exhibit convergence properties.[4] However, classical definitions of output stability require that the output space is a metric space, so that the distance between any *two* outputs is defined. This can be overkill when only convergence to a point or a set in the output space is considered.

Therefore, in this thesis, we use a more general notion of stability, accounting for the convergence of the trajectories with respect to a *cost function*, which can be any nonnegative function defined on the state space; see also Example 1.4 below. This definition allows for great flexibility in the notion of stability; in particular, it includes the concepts of stability discussed above. On top of this, it also allows to formulate global stability results taking into account the geometry of the state space in a concise and meaningful way (e.g., using a cost function that goes unbounded near the boundary of the state space).

Definition 1.11 (Cost function)**.** *Given a set* $X \subseteq \mathbb{R}^n$*, a* cost function on X *is any nonnegative scalar function* $\mathfrak{C} : X \to \mathbb{R}_{\geq 0}$.

Example 1.4. Examples of cost functions on $X \subseteq \mathbb{R}^n$ are $x \mapsto \|x-a\|$ (distance to a point $a \in X$, to account for the convergence of the trajectories to the point a), $x \mapsto \inf_{a\in A} \|x-a\|$ (distance to a set $A \subseteq X$, to account for the convergence of the trajectories to the set A), or $x \mapsto \|Px\|$ (where P is the projection on a subset of the variables, to account for the convergence of these variables of the system to zero). See also Figure 1.5 for an illustration.

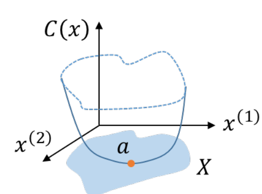

Figure 1.5: Example of a cost function that goes unbounded near the boundary of $X \subseteq \mathbb{R}^2$ and is zero at a single point $a \in X$.

The following notions, introduced by Hahn (1967), will be useful in the definitions of the concepts of stability, and in other places of this work.

Definition 1.12 (Class-$\mathcal{K}$ function)**.** *A function* $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is said to be of* class $\mathcal{K}$ *if it is continuous, strictly increasing and zero at zero.*

[4] For instance, for switched systems, it is generally sufficient that only the continuous variable ξ converges to zero while no convergence requirements are made on the switching signal σ.

Definition 1.13 (Class-$\mathcal{KL}$ function)**.** *A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of* class $\mathcal{KL}$ *if for every $t \in \mathbb{R}_{\geq 0}$, the function $r \mapsto \beta(r,t)$ is of class $\mathcal{K}$, and for every $r \in \mathbb{R}_{\geq 0}$, the function $t \mapsto \beta(r,t)$ is non-increasing and satisfies $\lim_{t\to\infty} \beta(r,t) = 0$.*

We also introduce the following definition, which will be useful to compare stability results with different cost functions.

Definition 1.14 ($\mathcal{K}$-equivalent functions)**.** *Let X be a set. Two functions $V_1 : X \to \mathbb{R}_{\geq 0}$ and $V_2 : X \to \mathbb{R}_{\geq 0}$ are* $\mathcal{K}$-equivalent *(or* equivalent*) if there are two class-$\mathcal{K}$ functions α_1 and α_2 such that for all $r \in X$, $V_1(r) \leq \alpha_1(V_2(r))$ and $V_2(r) \leq \alpha_2(V_1(r))$.*

It is readily seen that $\mathcal{K}$-equivalence defines an equivalence relation on the set of nonnegative functions defined on a given set.

Stability and feedback stabilization

First, we define the notion of stability, which applies to autonomous hybrid systems and accounts for the convergence of the cost of the trajectories to zero.

Definition 1.15 (Stability)**.** *Consider an autonomous hybrid system* $\mathsf{HySys} = (X, X_0, U, F, G)$ *and a cost function $\mathfrak{C} : X \to \mathbb{R}_{\geq 0}$.* HySys *is said to be* asymptotically stable *(or* stable*) with respect to $\mathfrak{C}$ if all trajectories of* HySys *are complete and there is a class-$\mathcal{KL}$ function β such that every trajectory ϕ of* HySys *satisfies*

$$\mathfrak{C}(\phi(t)) \leq \beta(\mathfrak{C}(\phi(0-)), t) \qquad \text{for all } t \in \mathbb{R}_{\geq 0}.$$

Proposition 1.16. *Consider an autonomous hybrid system* $\mathsf{HySys} = (X, X_0, U, F, G)$ *and two equivalent cost functions $\mathfrak{C}_1 : X \to \mathbb{R}_{\geq 0}$ and $\mathfrak{C}_2 : X \to \mathbb{R}_{\geq 0}$. Then,* HySys *is stable with respect to $\mathfrak{C}_1$ if and only if it is stable with respect to $\mathfrak{C}_2$.*

Proof. See Appendix A.1.1. □

Then, we introduce the notions of feedback controller and feedback composition of hybrid systems.

Definition 1.17 (Controller)**.** *A* feedback controller *(or* controller*) for a hybrid system (X, X_0, U, F, G) is an ordered pair (HySys_c, K) where $\mathsf{HySys}_c = (X_c, X_{c0}, X, F_c, G_c)$ is a hybrid system with input space X and $K : X \times X_c \rightrightarrows U$ is the* actuator map*. The controller is said to be* static *if X_c is a singleton.*

Definition 1.18 (Feedback composition). *Given a hybrid system* $\mathsf{HySys} = (X, X_0, U, F, G)$ *and a controller* $\mathsf{Cont} = (\mathsf{HySys}_\mathrm{c}, K)$, $\mathsf{HySys}_\mathrm{c} = (X_\mathrm{c}, X_{\mathrm{c}0}, X, F_\mathrm{c}, G_\mathrm{c})$, *for* HySys, *the* closed-loop system *obtained from* HySys *and* Cont *(also called* feedback composition *of* HySys *and* Cont*), denoted by* $\mathsf{HySys}\|\mathsf{Cont}$*, is the autonomous hybrid system* $(X_\mathrm{f}, X_{\mathrm{f}0}, \{0\}, F_\mathrm{f}, G_\mathrm{f})$ *where* $X_\mathrm{f} = X \times X_\mathrm{c}$*,* $X_{\mathrm{f}0} = X_0 \times X_{\mathrm{c}0}$*, and for* $\square \in \{F, G\}$*,* $\square_\mathrm{f}(x, x_\mathrm{c}, 0) = \bigcup_{u \in K(x,x_\mathrm{c})} \square(x, u) \times \square_\mathrm{c}(x_\mathrm{c}, x)$ *for all* $(x, x_\mathrm{c}) \in X \times X_\mathrm{c}$*.*

See Figure 1.6 for an illustration.

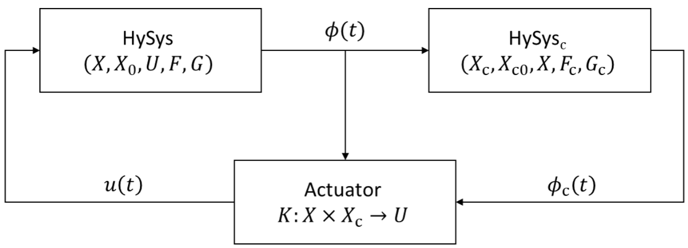

Figure 1.6: Feedback composition of a hybrid system HySys and a controller $\mathsf{Cont} = (\mathsf{HySys}_\mathrm{c}, K)$. If the controller is static, then the actuator takes only $\phi(t)$ as input.

Feedback composition can be used, among others, to obtain stable closed-loop systems from certain unstable hybrid systems, referred to as feedback stabilizable systems.

Definition 1.19 (Stabilizability). *Consider a hybrid system* $\mathsf{HySys} = (X, X_0, U, F, G)$ *and a cost function* $\mathfrak{C} : X \to \mathbb{R}_{\geq 0}$*.* HySys *is said to be* feedback stabilizable *(or* stabilizable*) with respect to* $\mathfrak{C}$ *if there is a controller* $\mathsf{Cont} = (\mathsf{HySys}_\mathrm{c}, K)$, $\mathsf{HySys}_\mathrm{c} = (X_\mathrm{c}, X_{\mathrm{c}0}, X, F_\mathrm{c}, G_\mathrm{c})$*, for* HySys *such that the closed-loop system* $\mathsf{HySys}\|\mathsf{Cont}$ *is stable with respect to the cost function* $\mathfrak{C}_\mathrm{f} : X \times X_\mathrm{c} \to \mathbb{R}_{\geq 0}$ *defined by* $\mathfrak{C}_\mathrm{f}(x, x_\mathrm{c}, 0) = \mathfrak{C}(x)$*.*

1.2 Smooth dynamical systems theory

Smooth dynamical systems arise as solutions of ordinary differential equations or difference equations with smooth vector field or smooth transition map (see Definitions 1.5 and 1.6 in Subsection 1.1.2). Smooth dynamical systems are deterministic and their trajectories depend continuously on the initial condition. They can nevertheless exhibit very rich behaviors and be very difficult to analyze. In this section, we review some of the tools for the analysis of these systems. In particular, we will remind the notions of generator, invariant sets, limit sets and Lyapunov functions for smooth dynamical systems. We will also discuss some more advanced topics, namely regarding the linearization and the structural stability (also called robustness) of these systems.

The section is organized as follows. In Subsection 1.2.1, we remind the notions of generator, invariant sets and limit sets of smooth dynamical systems. In Subsection 1.2.2, we introduce the concept of Lyapunov functions for smooth dynamical systems and review some of their properties that will be relevant for this thesis. Finally, in Subsection 1.2.3, we review some concepts from the linearization theory of smooth dynamical systems. In particular, we introduce the concepts of prolonged systems, hyperbolicity and structural stability of these systems.

References. The literature on ordinary differential equations and smooth dynamical systems is vast. The topic covers several centuries of major theoretical and practical advances (it goes back for instance to the works of Newton on the motion of celestial bodies) and still continues to be an active area of research; see, e.g., Katok and Hasselblatt (1995), Robinson (1999), Khalil (2002) and Teschl (2012) for modern treatments. Our principal reference for this section is Teschl (2012) covering most of the concepts and results discussed in this section, and also Robinson (1999) for the definition and properties of hyperbolic systems.

Notation. In this section, all considered hybrid systems are smooth dynamical systems, and thus for the sake of brievety, we will refer to them simply as *dynamical systems*. Also, we will consider both continuous-time systems and discrete-time systems, and we will use the symbol $\mathbb{T}$ to denote the time domain of the system, as it should be clear from the context whether $\mathbb{T} = \mathbb{R}$ (continuous-time systems) or $\mathbb{T} = \mathbb{Z}$ (discrete-time systems). We remind that, unless said otherwise, all considered trajectories of hybrid systems are assumed to be maximal.

1.2.1 Generator, invariant sets and limit sets

Dynamical systems were introduced in Subsection 1.1.2. For a reminder, the trajectories $\xi : \mathbb{T}_{\geq 0} \to \mathbb{R}^n$ of a dynamical system $\mathsf{Sys} = (\mathbb{R}^n, f)$ satisfy $\dot{\xi}(t) = f(\xi(t))$ for all $t \in \mathbb{R}_{\geq 0}$ (if Sys is continuous-time) or $\xi(t+1) = f(\xi(t))$ for all $t \in \mathbb{N}$ (if Sys is discrete-time), where $f : \mathbb{R}^n \to \mathbb{R}^n$ is infinitely differentiable (and Lipschitz continuous in the continuous-time case).

Dynamical systems have the property that the trajectories are *complete*. Moreover, from any given initial point, there is a *unique* trajectory starting at this point (see, e.g., Teschl, 2012, Theorem 2.17). This leads to the concept of generator of the trajectories of a dynamical system.

Definition 1.20 (Generator of a dynamical system)**.** *The* generator *of a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$*, denoted by* $\chi(\cdot, \cdot; \mathsf{Sys})$ *(or* χ *when* Sys *is clear from the context), is the function* $\chi : \mathbb{T}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ *defined by* $t \mapsto \chi(t, x)$ *is the unique trajectory of* Sys *with* $\chi(0, x) = x$*.*

The generator satisfies the following properties, which sometimes consist in the defining axioms of a dynamical system in the abstract theory of dynamical systems (see, e.g., Sontag, 1998).

Proposition 1.21 (Properties of the generator)**.** *Consider a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$*. The* generator χ *of* Sys *is infinitely differentiable in its second argument and satisfies the following property (called* semigroup axiom *by Sontag, 1998): for every* $t_0, t_1 \in \mathbb{T}_{\geq 0}$*,* $t_1 \geq t_0$*, and* $x \in \mathbb{R}^n$*,* $\chi(t_1, x) = \chi(t_1 - t_0, \chi(t_0, x))$*.*

Proof. See, e.g., Teschl (2012, Theorems 2.10 and 6.1) for the case of continuous-time systems. The case of discrete-time systems is similar. □

Given a dynamical system $\mathsf{Sys} = (\mathbb{R}^n, f)$, if for every $t \in \mathbb{T}_{\geq 0}$ and $x \in \mathbb{R}^n$, there is a unique $y \in \mathbb{R}^n$ such that $\chi(t, y) = x$, and y depends on x in an infinitely differentiable fashion, then we say that Sys is *invertible*. In this case, we extend the definition of χ on $\mathbb{T}_{<0} \times X$ by letting for each $(t, x) \in \mathbb{T}_{<0} \times X$, $\chi(t, x)$ be the unique point in X satisfying $\chi(-t, \chi(t, x)) = x$. By construction, the extended generator $\chi : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ also satisfies the properties of Proposition 1.21.

Now, we introduce the concept of invariance for subsets of the state space of dynamical systems; see also Figure 1.7 for an illustration.

Definition 1.22 (Set invariance)**.** *Consider a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$ *and a set* $\Lambda \subseteq \mathbb{R}^n$*. The set* Λ *is said to be* forward invariant *for* Sys *if for any* $x \in \Lambda$*,* $\chi(t, x) \in \Lambda$ *for all* $t \in \mathbb{T}_{\geq 0}$*. If* Sys *is invertible, the set* Λ *is said to be* backward invariant *for* Sys *if for any* $x \in \Lambda$*,* $\chi(t, x) \in \Lambda$ *for all* $t \in \mathbb{T}_{\leq 0}$*; and is said to be* invariant *for* Sys *if it is forward and backward invariant.*

The above invariance properties are closed under intersection and union. This leads to the notions of minimal and maximal invariant sets.

Proposition 1.23 (Minimal and maximal invariant sets)**.** *Consider an (invertible) dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$ *and a set* $\Lambda \subseteq \mathbb{R}^n$*. The union of all (forward, backward) invariant sets for* Sys *included in* Λ *is itself a (forward, backward) invariant set, called the* maximal (forward, backward) invariant set *of* Sys *in* Λ*. The intersection of all (forward, backward) invariant sets for* Sys *containing* Λ *is itself a (forward, backward) invariant set, called the* minimal (forward, backward) invariant set *of* Sys *containing* Λ*.*

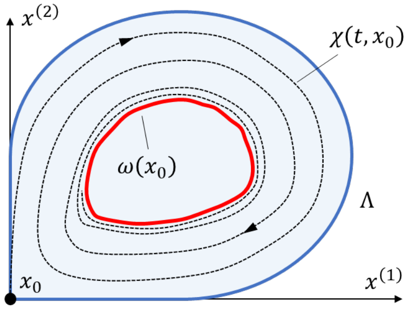

Figure 1.7: Invariant set. The set Λ (in blue) is forward invariant for Sys if every trajectory of Sys starting in Λ stays in Λ for all $t \geq 0$. In this image, the trajectory starting from x_0 accumulates on the set in red, called the ω-limit set of Sys from x_0 (see Definition 1.24).

Proof. See, e.g., Teschl (2012, Lemma 6.4) for the case of continuous-time systems. The case of discrete-time systems is similar. □

Limit sets are special types of invariant sets consisting in the sets of points in the state space where the trajectories of a given dynamical system eventually accumulate. They appear thus naturally in the study of the asymptotic properties of these systems (see, e.g., Subsections 1.2.2 and 2.3.3).

Definition 1.24 (ω-limit set)**.** *Consider a dynamical system* Sys $= (\mathbb{R}^n, f)$ *and a point* $x \in \mathbb{R}^n$*. The* ω-limit set *(or* limit set*) of* Sys *from* x*, denoted by* $\omega(x; \mathsf{Sys})$ *(or* $\omega(x)$ *when* Sys *is clear from the context), is the set of all points* $y \in \mathbb{R}^n$ *such that there is a sequence of times* $(t_j)_{j=1}^{\infty} \subseteq \mathbb{T}_{\geq 0}$ *satisfying* $\lim_{j\to\infty} t_j = \infty$ *and* $\lim_{j\to\infty} \chi(t_j, x) = y$*.*

The following properties of limit sets are elementary but useful for the study of the asymptotic behavior of dynamical systems.

Proposition 1.25 (Properties of limit sets)**.** *Consider a dynamical system* Sys $= (\mathbb{R}^n, f)$ *and a point* $x \in \mathbb{R}^n$*.*

- *The limit sets of* Sys *are closed and forward invariant. If* Sys *is invertible, then its limit sets are also backward invariant (hence, invariant).*

- *If* $\{\chi(t, x) : t \in \mathbb{T}_{\geq 0}\}$ *(called the* orbit *of* Sys *from* x*) is contained in a compact subset of* $\mathbb{R}^n$*, then* $\omega(x)$ *is nonempty and compact; moreover, in that case,* $\chi(\cdot, x)$ *converges to* $\omega(x)$*, in the sense that* $\lim_{t\to\infty} \inf_{y\in\omega(x)} \|\chi(t, x) - y\| = 0$*.*

Proof. See, e.g., Teschl (2012, Lemmas 6.5–6.7) for the case of continuous-time systems. The case of discrete-time systems is similar. □

1.2.2 Lyapunov functions and Lyapunov theory

We start with the definition of a Lyapunov function. A Lyapunov function can be seen as an "energy" function that decreases along the trajectories of the system; a bit like a ball in a valley sees its mechanical energy decrease with time, until it stops at the bottom of the valley (state at which its mechanical energy is the lowest); see also Figure 1.8 for an illustration.

Definition 1.26 (Lyapunov function)**.** *Consider a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$*. A* Lyapunov function *for* Sys *is a function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *satisfying that for all* $x \in \mathbb{R}^n$ *and* $t \in \mathbb{T}_{\geq 0}$*,* $V(\chi(t, x)) \leq V(x)$*, and there is* $T \in \mathbb{T}_{>0}$ *and a class-*$\mathcal{K}$ *function* α *such that for all* $x \in \mathbb{R}^n$*,* $V(\chi(T, x)) - V(x) \leq -\alpha(V(x))$*.*

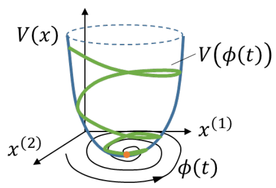

Figure 1.8: Lyapunov function V for a dynamical system Sys. The value of V (in green) along the trajectory ϕ (in black) decreases as time evolves.

Just like a ball starting at rest in a valley will never go higher than its initial height, sublevel sets of Lyapunov functions define forward invariant sets, in the sense that the trajectories starting in such a sublevel set cannot escape from it. This is known as the *Krasovskii–LaSalle principle.*

Proposition 1.27 (Krasovskii–LaSalle principle)**.** *Consider a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$ *and let* V *be a Lyapunov function for* Sys*. Then, for any* $c \in \mathbb{R}$*, the sublevel sets* $\{x \in \mathbb{R}^n : V(x) \leq c\}$ *and* $\{x \in \mathbb{R}^n : V(x) < c\}$ *are forward invariant for* Sys*. Furthermore,* V *is zero on any limit set of* Sys*.*

Proof. See, e.g., Teschl (2012, Theorem 6.15) for the case of continuous-time systems. The case of discrete-time systems is similar. □

Lyapunov functions appear naturally in the study of the asymptotic behavior of dynamical systems. Indeed, Propositions 1.27 and 1.25 imply that the trajectories of the system either converge towards the exterior of the state space, or they converge to a set of points on which V is zero (see also, e.g., Khalil, 2002, Theorem 4.4, for a proof with slightly different assumptions on the Lyapunov function). The above argument can be further refined to conclude that the system is stable with respect to the cost function induced by the Lyapunov function.

Proposition 1.28 (Stability from Lyapunov function)**.** *Consider a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$ *and a cost function* $\mathfrak{C} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$*. Let* V *be a Lyapunov* *function for* Sys *and assume that* V *is equivalent to* $\mathfrak{C}$*. Then,* Sys *is stable with respect to* $\mathfrak{C}$*.*

Proof. See Appendix A.1.2. □

In particular, if V is radially unbounded and zero only at the origin, then the system is stable with respect to the classical cost $x \mapsto \|x\|$ (see, e.g., Khalil, 2002, Theorems 4.9 and 4.10, for a proof with slightly different assumptions on the Lyapunov function).

Lyapunov functions have been an important field of research in dynamical systems theory for some time now (see, e.g., Hahn, 1967, and Khalil, 2002, for surveys). In particular, the questions of existence and computation of Lyapunov functions for stable dynamical systems have received a lot of attention from the systems and control community; under the name of *converse Lyapunov theorems* and *Lyapunov's direct method* respectively. We refer the reader to the references above for surveys on converse Lyapunov theorems and Lyapunov's direct method.

1.2.3 Linearization theory

We start with the definition of the prolonged system, introduced by Crouch and van der Schaft (1987).

Definition 1.29 (Prolonged system)**.** *Given a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$*, the* prolonged system *(also called the* linear extension*) of* Sys*, denoted by* $\partial\mathsf{Sys}$*, is the dynamical system defined by* (X_∂, f_∂) *where* $X_\partial = \mathbb{R}^n \times \mathbb{R}^n$ *and* $f_\partial(x, v) = \big(f(x), \frac{\partial f}{\partial x}(x)v\big)$ *for all* $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$*.*

In the above definition, $\frac{\partial f}{\partial x} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ stands for the derivative (also called *Jacobian matrix*) of f. From the definition, the trajectories $(\xi, \delta) : \mathbb{T}_{\geq 0} \to \mathbb{R}^n \times \mathbb{R}^n$ of the prolonged system satisfy

$$\begin{cases} \dot{\xi}(t) = f(\xi(t)) \\ \dot{\delta}(t) = \frac{\partial f}{\partial x}(\xi(t))\delta(t) \end{cases} \qquad \text{(continuous-time)},$$

$$\begin{cases} \xi^+(t) = f(\xi(t)) \\ \delta^+(t) = \frac{\partial f}{\partial x}(\xi(t))\delta(t) \end{cases} \qquad \text{(discrete-time)},$$

for all $t \in \mathbb{T}_{\geq 0}$.

The prolonged system accounts for the *sensitivity of the trajectories* of a dynamical system *to the initial condition*, defined as the derivative of the generator χ with respect to its second argument; see also Figure 1.9 for an illustration.

Proposition 1.30 (Sensitivity to initial condition)**.** *Consider a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$*. Then, for any* $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$*, the function* $t \mapsto \big(\chi(t, x), \frac{\partial \chi}{\partial x}(t, x)v\big)$ *is a trajectory of the prolonged system* $\partial\mathsf{Sys}$*.*

Proof. See, e.g., Teschl (2012, Theorem 2.10) for the case of continuous-time systems. The case of discrete-time systems is similar. □

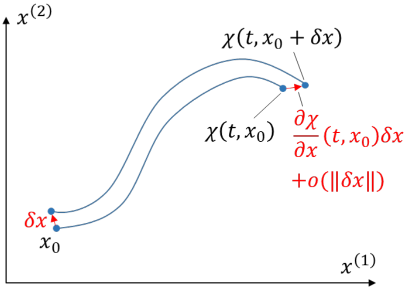

Figure 1.9: Sensitivity of trajectories to the initial condition. The displacement of the initial condition x_0 by a small δx translates as the displacement of the state at time t by $\frac{\partial \chi}{\partial x}(t, x_0)\delta x$.

The study of linearized dynamics was introduced by Lyapunov (1892) for the analysis of the local stability of fixed points of dynamical systems. Indeed, given a dynamical system $\mathsf{Sys} = (\mathbb{R}^n, f)$ and a fixed point $a \in \mathbb{R}^n$ for Sys, it holds that if the Jacobian matrix $\frac{\partial f}{\partial x}$ at a provides a stable linear system, then a is locally stable, meaning that there is a neighborhood A of a that is forward invariant for Sys and such that the system restricted to A is stable with respect to the classical cost $x \mapsto \|x - a\|$ (see, e.g., Khalil, 2002, Theorem 4.13, or Robinson, 1999, Theorems 5.5.1 and 5.6.1). This approach is sometimes referred to as *Lyapunov's indirect method* for the stability analysis of fixed points of dynamical systems. Note that the above result can be generalized for the analysis of the *incremental stability* (also called *contraction analysis*) of dynamical systems, using the prolonged system (see, e.g., Lohmiller and Slotine, 1998, and Forni and Sepulchre, 2014).

The prolonged system can also be used to analyze the global behavior of the system, for instance to ensure the existence of simple attractors for the system (see, e.g., Forni and Sepulchre, 2016, and Forni and Sepulchre, 2019) or the structural stability of its invariant sets, as explained in the subsubsection below. See also Section 2.3 for applications in the context of dominance analysis of dynamical systems.

Hyperbolicity and structural stability

In the preamble of this thesis, we introduced the concept of hyperbolicity, which describes the property that the linearized dynamics of a dynamical system can be split into an unstable component and a stable component.

In this sense, the property of hyperbolicity generalizes the notion of *hyperbolic fixed point.* Indeed, a hyperbolic fixed point of a dynamical system $\mathsf{Sys} = (\mathbb{R}^n, f)$ is a fixed point a of Sys such that the Jacobian matrix $\frac{\partial f}{\partial x}(a)$ at a has no *marginally stable* eigenvalues. Hence, since the prolonged system $\partial\mathsf{Sys}$ around a is described by the linear dynamical system with matrix $\frac{\partial f}{\partial x}(a)$, the linearized dynamics of Sys around a can be split into components that are either stable or unstable (and never marginally stable); see also Figure 1.10 for an example. Hyperbolic fixed points have the property of being *structurally stable* (aka. *robust to model perturbations*), meaning that the qualitative behavior of the system around the fixed point is not affected by small perturbations of the system: indeed, it is well known from bifurcation theory that a fixed point a of a dynamical system can appear or disappear, or change its nature (stable vs. unstable) only if the Jacobian matrix at a has a marginally stable eigenvalue (see, e.g., Robinson, 1999, Theorem 5.6.4).

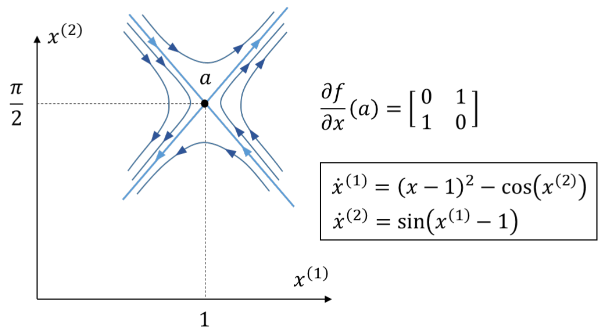

Figure 1.10: Hyperbolic fixed point of a continuous-time dynamical system $\mathsf{Sys} = (\mathbb{R}^2, f)$. The point $a = (1, \pi/2)$ is a fixed point for the system described on the right of the picture. The Jacobian of f at a has two eigenvalues, 1 and -1, which are not marginally stable. The *Hartman–Grobman* theorem implies that the trajectories of the system near the hyperbolic fixed point are close to those of the linear system given by the Jacobian of f at a (see, e.g., Teschl, 2012, Theorem 9.9).

The concept of hyperbolic fixed point generalizes to invariant sets of *invertible* dynamical systems $\mathsf{Sys} = (\mathbb{R}^n, f)$ as follows (see also Definition 1.31 below). For every point x in the invariant set, the state space of the linearized system (i.e., $\mathbb{R}^n$) can be split into two complementary subspaces, denoted by $E^{\mathrm{s}}(x)$ and $E^{\mathrm{u}}(x)$, called the *stable* and *unstable subspaces* at x. The function associating these subspaces to each point of the invariant set is sometimes called a *hyperbolic splitting*, and it has the property of being invariant for the prolonged system $\partial\mathsf{Sys}$ and satisfies that the linearized dynamics of Sys starting from the subspace $E^{\mathrm{s}}(x)$ (resp. E^{u}) converges exponentially to zero as t goes to ∞ (resp. $-\infty$; remember that Sys is invertible). This is formalized in the following definition.

Definition 1.31 (Hyperbolic invariant set). *Consider an invertible dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$ *and an invariant set* $\Lambda \subseteq \mathbb{R}^n$ *for* Sys. *The set* Λ *is said to be* hyperbolic *for* Sys *if there are two set-valued functions* $E^{\mathrm{s}} : \Lambda \rightrightarrows \mathbb{R}^n$ *and* $E^{\mathrm{u}} : \Lambda \rightrightarrows \mathbb{R}^n$ *such that (i) for every* $x \in \Lambda$, $E^{\mathrm{s}}(x)$ *and* $E^{\mathrm{u}}(x)$ *are linear subspaces satisfying* $\mathbb{R}^n = E^{\mathrm{s}}(x) \oplus E^{\mathrm{u}}(x)$, *(ii)* E^{s} *and* E^{u} *are invariant under the action of the prolonged system in the sense that for all* $x \in \Lambda$ *and* $t \in \mathbb{T}$, $\frac{\partial \chi}{\partial x}(t,x)E^{\diamond}(x) = E^{\diamond}(\chi(t,x))$ *for* $\diamond \in \{\mathrm{u},\mathrm{s}\}$, *and (iii) there is* $\lambda < 0$ *and* $C \geq 0$ *such that for all* $(x,v) \in \Lambda \times \mathbb{R}^n$,

- *if* $v \in E^{\mathrm{s}}(x)$, *then* $\|\frac{\partial \chi}{\partial x}(t,x)v\| \leq C\|v\|e^{\lambda|t|}$ *for all* $t \in \mathbb{T}_{\geq 0}$;
- *if* $v \in E^{\mathrm{u}}(x)$, *then* $\|\frac{\partial \chi}{\partial x}(t,x)v\| \leq C\|v\|e^{\lambda|t|}$ *for all* $t \in \mathbb{T}_{\leq 0}$.

Remark 1.5. The requirements (ii) and (iii) in Definition 1.31 imply that the functions E^{s} and E^{u} are unique and depend continuously on their argument (see, e.g., Robinson, 1999, Remark 8.1.6). This implies that the dimensions of E^{s} and E^{u} are constant on every connected component of Λ.

The property of structural stability of hyperbolic fixed points extends to hyperbolic invariant sets, as first noticed by Anosov (1967) and Smale (1967) for the so-called *Anosov systems* (systems that are hyperbolic on their whole domain) and generalized by Hirsch and Pugh (1970) to isolated hyperbolic invariant sets.

Definition 1.32 (Isolated invariant set). *Consider an invertible dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$. *A closed set* $\Lambda \subseteq \mathbb{R}^n$ *invariant for* Sys *is said to be* isolated *if there is a neighborhood* U *of* Λ *such that* Λ *is the maximal invariant set of* Sys *contained in* U. *Such a set* U *is called an* isolating neighborhood *for* Λ.

Proposition 1.33 (Structural stability of hyperbolic invariant sets). *Consider an invertible dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$, *and let* $\Lambda \subseteq \mathbb{R}^n$ *be a hyperbolic isolated invariant set for* Sys, *with isolating neighborhood* U. *Then,* Λ *is* structurally stable *(also called* robust to model perturbations*) for* Sys*: this means that there is* $\epsilon > 0$ *such that any dynamical system* $\mathsf{Sys}' = (\mathbb{R}^n, f')$, *with* f' ϵ*-close to* f *in the* $C^1(U)$*-topology*[5], *has a nonempty maximal invariant set* Λ' *in* U *and satisfies that* $f|_{\Lambda}$ *and* $f|_{\Lambda'}$ *are* topologically conjugate[6].

[5] This means that $\sup_{x\in U}\|f(x) - f'(x)\| \leq \epsilon$ and $\sup_{x\in U}\|\frac{\partial f}{\partial x}(x) - \frac{\partial f'}{\partial x}(x)\| \leq \epsilon$.

[6] This means that there is a homeomorphism (that is, a bijective bi-continuous function) $h : \Lambda \to \Lambda'$ such that $\chi(t,x;\mathsf{Sys}')) = h^{-1}(\chi(t,h(x);\mathsf{Sys}))$ for all $x \in \Lambda'$ and $t \in \mathbb{T}$.

Proof. See, e.g., Robinson (1999, Theorem 10.7.4) for the case of discrete-time systems. The case of continuous-time systems can be deduced by time discretization. □

As for hyperbolic fixed points, it can be shown that hyperbolicity is in general not only a sufficient condition but also a necessary condition for structural stability. Indeed, under some mild assumptions (e.g., *Axiom A*, or *no-cycle condition*, etc.), the structurally stable dynamical systems are precisely the ones that are hyperbolic on some distinguished sets (e.g., limit set, *chain-recurrent set*, etc.; see, e.g., Robinson, 1999, Chapters 10 and 11).

First developed for invertible dynamical systems (as in Definition 1.31), hyperbolicity has rapidly become a cornerstone of dynamical systems theory and found applications in many different areas, like chaos, ergodic theory, entropy, etc. (see, e.g., Robinson, 1999, and Hasselblatt, 2017, for comprehensive surveys of results related to hyperbolic dynamics). Hyperbolicity has also been generalized in several directions, accounting for various classes of systems while retaining the main features of hyperbolic dynamics: see, e.g., Hasselblatt and Pesin (2006) (partial hyperbolicity), Barreira and Pesin (2006) (non-uniform hyperbolicity), Berger and Rovella (2013) (non-invertible hyperbolic dynamical systems), Colonius and Du (2001) (hyperbolic control systems).

1.3 Switched systems theory

Switched systems are hybrid systems described by a finite set of continuous dynamics among which the system can switch over time (see Definitions 1.7 and 1.8 in Subsection 1.1.2). These systems arise naturally in the modeling of phenomenons, processes or devices presenting sudden changes between different modes of operation. By restricting the set of admissible transitions of the system between the different modes, one obtains *constrained switched systems*. This allows for instance to increase the expressiveness of these systems by considering only the transitions that happen in practice; see, e.g., Example 1.3. Constrained switching can also be used as a design parameter to ensure that the system satisfies some properties. The function describing the current mode of the system is called the switching signal (see Subsection 1.1.2). The restrictions on the transitions of the system between the different modes can thus be enforced as constraints on the switching signal.

In this section, we discuss two types of such constraints on the switching signal, namely switching with dwell time and switching constrained by a timed automaton. We also review some of the tools for the analysis of switched systems. In particular, we discuss the notions of stability for switched linear systems and we introduce the concepts of multiple Lyapunov functions and path-complete Lyapunov functions, which extend the theory of Lyapunov functions to switched systems.

The section is organized as follows. In Subsection 1.3.1, we introduce the notions of generator of switched systems, and we discuss different types of constraints on the switching signal. In Subsection 1.3.2, we introduce switched linear systems and discuss the notions of stability for these systems. In particular, we introduce the concept of joint spectral radius of switched linear systems under arbitrary switching and discuss some of its properties that will be relevant for this work. Finally, in Subsection 1.3.3, we introduce the concepts of multiple and path-complete Lyapunov functions and review some of their properties for the stability analysis of (constrained) switched linear systems.

References. As paradigmatic examples of hybrid systems, switched systems have received a lot of attention in the literature; see, e.g., Liberzon (2003), Lin and Antsaklis (2009), Jungers (2009) and Sun and Ge (2011) for introductions. Our main references for this section are Liberzon (2003) and Sun and Ge (2011) covering most of the concepts and results discussed in this section, and Jungers (2009) for the definition and properties of the joint spectral radius.

Notation. We will consider both continuous-time systems and discrete-time systems, and we will use the symbol $\mathbb{T}$ to denote the time domain of the system, as it should be clear from the context whether $\mathbb{T} = \mathbb{R}$ (continuous-time systems) or $\mathbb{T} = \mathbb{Z}$ (discrete-time systems). We remind that, unless said otherwise, all considered trajectories of hybrid systems are assumed to be maximal.

1.3.1 Generator and switching signals

In Subsection 1.1.2, we saw that a switched system can be described by the triple $(\mathbb{R}^n, U, \{f_i\}_{i\in\Sigma})$ where $\mathbb{R}^n$ is the state space of the continuous variable, U is the input space and for each $i \in \Sigma$, $f_i : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a continuous (even Lipschitz continuous in its first argument in the continuous-time case) function, and possibly additional assumptions on the right-continuous, piecewise constant[7] switching signal. Therefore, given a switched system SwS, we will use the notation $\mathsf{SwS} \sim (\mathbb{R}^n, U, \{f_i\}_{i\in\Sigma})$ to mean that SwS has state space $\mathbb{R}^n$ for the continuous variable, input space U, set of functions $\{f_i\}_{i\in\Sigma}$, and possibly additional assumptions on the switching signals (typically spec-

[7] We saw in Subsection 1.1.2 that the switching signals of a continuous-time switched system are right-continuous and piecewise constant. This also holds for discrete-time switched systems since functions defined on a discrete set are trivially right-continuous and piecewise constant.

ified at the introduction of SwS). For a reminder, the complete trajectories $(\xi, \sigma) : \mathbb{T}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ of a switched system $\mathsf{SwS} \sim (\mathbb{R}^n, U, \{f_i\}_{i \in \Sigma})$ with input $u : \mathbb{T}_{\geq 0} \to \mathbb{R}^n$ satisfy $\dot{\xi}(t) = f_{\sigma(t)}(\xi(t), u(t))$ for all $t \in \mathbb{R}_{\geq 0}$ (if SwS is continuous-time) and $\xi(t+1) = f_{\sigma(t)}(\xi(t), u(t))$ for all $t \in \mathbb{N}$ (if SwS is discrete-time), where $\xi : \mathbb{T}_{\geq 0} \to \mathbb{R}^n$ is the continuous variable and $\sigma : \mathbb{T}_{\geq 0} \to \Sigma$ is the switching signal of the trajectory.

Definition 1.34 (Set of complete switching signals)**.** *Given a switched system* SwS*, we denote by* $\mathcal{S}(\mathsf{SwS})$ *(or* $\mathcal{S}$ *if* SwS *is clear from the context) the set of* complete switching signals *(i.e., switching signals corresponding to complete trajectories) of* SwS*.*

We restrict our attention to the set of *complete* switching signals since, by definition of switched systems (see Definitions 1.7 and 1.8 in Subsection 1.1.2), the switching signals are the trajectories of an autonomous hybrid subsystem. Hence, non-complete switching signals are in general not interesting as it implies that the trajectory of the switched system stops only because the switching signal cannot be continued.

The set of complete switching signals of a switched system has the following *shift-invariance* property.

Proposition 1.35 (Forward shift-invariance of $\mathcal{S}$)**.** *Consider a switched system* SwS *with set of mode* Σ*. The set* $\mathcal{S}$ *of complete switching signals of* SwS *is* closed under forward time shift*, meaning that for all* $\sigma \in \mathcal{S}$ *and* $t_0 \in \mathbb{T}_{\geq 0}$*, the signal* $\sigma' : \mathbb{T}_{\geq 0} \to \Sigma$ *defined by* $\sigma'(t) = \sigma(t + t_0)$ *satisfies* $\sigma' \in \mathcal{S}$*.*

Proof. Straightforward from the definition of the switching signal as the projection of the trajectory of a hybrid system. □

The assumptions on the set of functions of a switched system $\mathsf{SwS} \sim (\mathbb{R}^n, U, \{f_i\}_{i \in \Sigma})$ imply that for any input $u \in \mathcal{U}$, any switching signal $\sigma \in \mathcal{S}$ and any initial point $x \in \mathbb{R}^n$, there is a *unique* function $\xi : \mathbb{T}_{\geq 0} \to \mathbb{R}^n$ such that (ξ, σ) is a trajectory of SwS with input u and with $\xi(0) = x$. This leads to the concept of generator of the trajectories of a switched system.

Definition 1.36 (Generator of a switched system)**.** *The* generator *of a switched system* $\mathsf{SwS} \sim (\mathbb{R}^n, U, \{f_i\}_{i \in \Sigma})$*, denoted by* $\chi(\cdot, \cdot, \cdot, \cdot, \cdot; \mathsf{SwS})$ *(or* χ *when* SwS *is clear from the context), is the function* $\chi : \mathcal{T} \times \mathbb{R}^n \times \mathcal{S} \times \mathcal{U} \to \mathbb{R}^n$ *with* $\mathcal{T} = \{(t_1, t_0) \in \mathbb{T}_{\geq 0} \times \mathbb{T}_{\geq 0} : t_1 \geq t_0\}$*, defined by* $\chi(t_1, t_0, x, \sigma, u) = \xi'(t_1 - t_0)$ *where* $(\xi', \sigma') : \mathbb{T}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ *is the trajectory of* SwS *starting at* x *with input* $u' : \mathbb{T}_{\geq 0} \to U$*, defined by* $\sigma'(t) = \sigma(t + t_0)$ *and* $u'(t) = u(t + t_0)$ *for all* $t \in \mathbb{T}_{\geq 0}$*.*

Remark 1.6. If $t_0 = 0$, we also use the shortened notation "$\chi(t, x, \sigma, u)$" to denote "$\chi(t, 0, x, \sigma, u)$". If SwS is autonomous, we use the shortened notation "$\chi(t, t_0, x, \sigma)$" to denote "$\chi(t, t_0, x, \sigma, u)$" where u is the unique function in $\mathcal{U}$.

The generator satisfies the following properties, which sometimes consist in the defining axioms of a control system in the abstract theory of control systems (see, e.g., Sontag, 1998).

Proposition 1.37 (Properties of the generator)**.** *Consider a switched system* $\mathsf{SwS} \sim (\mathbb{R}^n, U, \{f_i\}_{i\in\Sigma})$*. The* generator χ *of* SwS *is continuous in its third argument and satisfies the following property (called* semigroup axiom *by Sontag, 1998): for every* $t_0, t_1, t_2 \in \mathbb{T}_{\geq 0}$*,* $t_2 \geq t_1 \geq t_0$*,* $x \in \mathbb{R}^n$*,* $\sigma \in \mathcal{S}$ *and* $u \in \mathcal{U}$*, it holds that* $\chi(t_2, t_0, x, \sigma, u) = \chi(t_2, t_1, \chi(t_1, t_0, x, \sigma, u), \sigma, u)$*.*

Proof. See, e.g., Teschl (2012, Theorem 2.8) for a proof of the continuity with respect to the third argument, in the case of continuous-time systems. The case of discrete-time systems is similar. The proof of the second property is direct. □

Adding assumptions on the switching signals of switched systems allows to increase their expressiveness to model a wide range of complex systems and phenomenons. We discuss below different types of such assumptions that can be made on the switching signal of switched systems.

Arbitrary switching

When there is no assumption on the switching signal except that it is right-continuous and piecewise constant, we say that the switched system is under *arbitrary switching.*

For continuous-time switched systems under arbitrary switching, the requirement that the maximal trajectories are complete cannot be enforced in the definition of the system as a hybrid system. Nevertheless, the complete trajectories of a continuous-time switched system $\mathsf{SwS} \sim (\mathbb{R}^n, U, \{f_i\}_{i\in\Sigma})$ under arbitrary switching are connected to the trajectories of the associated differential inclusion.

Definition 1.38. *The* differential inclusion *associated to a continuous-time switched system* $\mathsf{SwS} \sim (\mathbb{R}^n, U, \{f_i\}_{i\in\Sigma})$ *under arbitrary switching is the hybrid system* $\mathsf{HySys} = (\mathbb{R}^n, U, F, G)$ *where* $F(x, u) = \{f_i(x, u)\}_{i\in\Sigma}$ *and* $G : \mathbb{R}^n \times U \rightrightarrows \emptyset$ *is the empty map.*

It holds that every trajectory of HySys with input $u \in \mathcal{U}$ is complete and can be approximated locally by a trajectory of SwS with the same input (see also Sun and Ge, 2011, Subsection 2.3.1, for a stronger result—stated without proof though—in the case of switched linear systems).

Proposition 1.39. *Let* $\mathsf{SwS} \sim (\mathbb{R}^n, U, \{f_i\}_{i\in\Sigma})$ *be a continuous-time switched system under arbitrary switching and* HySys *be the associated differential inclusion. Let* $\phi : \mathbb{R} \to \mathbb{R}^n$ *be a trajectory of* HySys *with input* $u \in \mathcal{U}$*. Then, for any* $\epsilon > 0$ *and* $T \in \mathbb{R}_{\geq 0}$*, there is a complete trajectory* (ξ, σ) *of* SwS *with input* u *such that for all* $t \in [0, T]$*,* $\|\phi(t) - \xi(t)\| \leq \epsilon$*.*

Proof (sketch). The proof follows from the fact that any L^1 function can be approximated arbitrarily well by a right-continuous, piecewise constant function in the L^1 topology (see, e.g., Friedman, 1982, Problem 3.2.2). The details are omitted since the study of differential inclusions is not the objective of this thesis. □

Furthermore, although the requirement that the maximal trajectories are complete cannot be explicitly enforced for continuous-time switched systems under arbitrary switching, it will be naturally satisfied when some other assumptions, which can be expressed in the formalism of hybrid systems, are made on the switching signal. We think for instance to *slow-switching* assumptions, such as dwell-time or average dwell-time assumptions, which are introduced below.

Remark 1.7. While it is interesting to see which types of assumptions on the switching signal can be expressed in the formalism of hybrid systems to appreciate the scope and flexibility of this formalism, and to keep a consistent and limited set of notations through this work, this should not be an imperative in itself and should not be a limitation for the theoretical and practical study of switched systems.

Dwell-time switching and constrained switching

The concept of *dwell time* was introduced by Morse (1996) to describe switching signals that require a minimal amount of time between two switches. This concept was later generalized by Hespanha and Morse (1999), under the name of *average dwell time*, to account for the property that the number of switches in a bounded interval grows at most linearly with the length of the interval. Both concepts have become standard in the study of switched systems (see, e.g., Liberzon, 2003, Section 3.2, and Goebel et al., 2012, Section 2.4).

Definition 1.40 (Dwell time and average dwell time)**.** *Consider a finite set* Σ *and a right-continuous, piecewise constant function* $\sigma : \mathbb{R} \to \Sigma$*. For* $t_1, t_0 \in \mathbb{R}$*,*

$t_1 \geq t_0$, let $N_\sigma(t_1, t_0)$ be the number of discontinuity points of σ in $[t_0, t_1)$. Let $\tau_\mathrm{a} > 0$ and $N_\circ \geq 0$.

- *We say that σ has* average dwell time *τ_a (with parameter $N_\circ$) if for any $t_1, t_0 \in \mathbb{R}$, $t_1 \geq t_0$, it holds that $N_\sigma(t_1, t_0) \leq N_\circ + \frac{t_1 - t_0}{\tau_\mathrm{a}}$.*
- *We say that σ has* absolute dwell time *(or* dwell time*) τ_a if it has average dwell τ_a with parameter $N_\circ = 1$ (this is equivalent to saying that there is at least τ_a units of time between any two discontinuities of σ).*

A function $\sigma : E \to \Sigma$, with $E \subseteq \mathbb{R}$, is said to have average dwell time *τ_a (with parameter $N_\circ$) (resp.* absolute dwell time *τ_a) if there is a right-continuous, piecewise constant extension of σ on $\mathbb{R}$ with average dwell time τ_a and with parameter $N_\circ$ (resp. absolute dwell time τ_a). For $t_1, t_0 \in \mathbb{R}$, $t_1 \geq t_0$, we let $N_\sigma(t_1, t_0) = \min \{N_{\hat{\sigma}}(t_1, t_0) : \hat{\sigma}$ is a right-continuous, piecewise constant extension of σ on $\mathbb{R}\}$.*

Definition 1.41 (Switched system with dwell time and average dwell time)**.** *Consider a switched system* SwS*, and let $\tau_\mathrm{a} > 0$ and $N_\circ \geq 0$.*

- *We way that* SwS *has* average dwell time *τ_a (with parameter $N_\circ$) if all its switching signals have average dwell time τ_a with parameter $N_\circ$.*
- *We say that* SwS *has* absolute dwell time *(or* dwell time*) τ_a if all its switching signals have dwell time τ_a.*

See Figure 1.11 for an illustration.

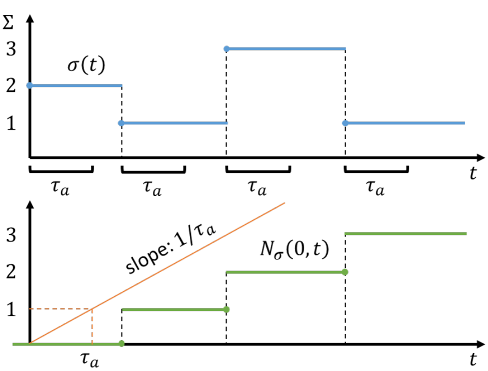

Figure 1.11: *Top:* Switching signal σ with set of modes $\Sigma = \{1, 2, 3\}$, and with absolute dwell time τ_a. *Bottom:* Number of switches of σ in the interval $[0, t)$. The number of switches is always smaller than t/τ_a since σ has absolute (and thus average) dwell time τ_a.

Remark 1.8. Interestingly, the condition that a switched system has an absolute dwell time or an average dwell time can be expressed in the formalism of hybrid systems (see, e.g., Goebel et al., 2012, Section 2.4).

On top of slow-switching assumptions (like absolute dwell-time and average dwell-time assumptions), it is sometimes useful to restrict also the transitions of the system between different modes. This allows for instance to use switched systems as abstractions of complex systems while retaining the main features of the original system (see, e.g., Philippe, 2017, and Section 2.3). The set of admissible transitions between modes can also be used as a design parameter to ensure that the switched system satisfies some properties (see, e.g., Philippe, 2017, and Gomes et al., 2018).

To describe the set of admissible transitions and dwell times between transitions, we use a *timed automaton*, which is a directed graph with edges labeled by symbols from a finite alphabet (typically the set of modes of the switched system) and by "dwell time sets" which are subsets of $\mathbb{R}_{\geq 0}$; see also Figure 1.12 for an illustration.

Definition 1.42 (Timed automaton)**.** *A* timed automaton *is a triplet* (Q, Σ, Θ) *where* Q *is a finite set, called the set of* states*,* Σ *is a finite set, called the* alphabet*, and* $\Theta \subseteq Q \times Q \times \Sigma \times 2^{\mathbb{R}_{\geq 0}}$ *is a finite set of* timed transitions*.*

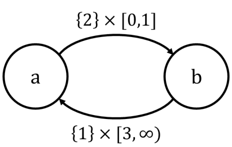

Figure 1.12: Timed automaton with two states $Q = \{\mathtt{a}, \mathtt{b}\}$ and two transitions $\Theta = \{(\mathtt{a}, \mathtt{b}, 2, [0, 1]), (\mathtt{b}, \mathtt{a}, 1, [3, \infty))\}$. This automaton accepts every switching signal that does not stay longer than 1 (unit of time) in mode 2 and stays at least 3 (units of time) in mode 1 (see Definition 1.43).

Remark 1.9. The term "timed automaton" is used here in a sense different than its usual meaning in automata theory (see, e.g., Mitra, 2021, Section 4.3.6). Classical timed automata are directed graphs together with a set of *clocks*, and each edge of the graph is associated with a set of *clock constraints* (specifying when the edge can be taken) and a set of *resets* (specifying which clocks are reset to zero when the edge is taken). The timed automata considered here correspond to classical timed automata with a single clock and with reset of this clock for each edge of the graph.

For a transition $\theta = (q_1, q_2, i, E) \in \Theta$, we denote its *source* q_1 by $\mathsf{s}(\theta)$, its *target* q_2 by $\mathsf{t}(\theta)$, its *mode* i by $\mathsf{i}(\theta)$, and its *set of durations* E by $\mathsf{d}(\theta)$. A *path* of length J, with $J \in \mathbb{N} \cup \{\infty\}$, in a timed automaton (Q, Σ, Θ) is any sequence $(\theta_j)_{j=0}^{J-1} \subseteq \Theta$ satisfying that $\mathsf{t}(\theta_j) = \mathsf{s}(\theta_{j+1})$ for all $j \in \{0, \ldots, J-2\}$.

Definition 1.43 (Admissible signal for a timed automaton)**.** *Consider a timed automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$*. A function* $\sigma : E \to \Sigma$*, with* $E \subseteq \mathbb{R}$*, is said to be* admissible *for* Aut *(or* accepted *by* Aut*) if* σ *has an extension* $\hat{\sigma}$ *on* $\mathbb{R}$ *satisfying* *that for any* $T_0, T_1 \in \mathbb{R}$, $T_1 \geq T_0$, *there is a finite sequence of times* $(\tau_j)_{j=0}^{J+1} \subseteq \mathbb{R}$, $T_0 = \tau_0 < \tau_1 < \ldots < \tau_{J+1} = T_1$, *and a path* $(\theta_j)_{j=0}^{J-1}$ *in* Aut *such that (i) for all* $j \in \{0, \ldots, J\}$, $\hat{\sigma}$ *is constant on* $[\tau_j, \tau_{j+1})$, *and (ii) for all* $j \in \{1, \ldots, J-1\}$, $\hat{\sigma}(\tau_j) = \mathsf{i}(\theta_j)$ *and* $\tau_{j+1} - \tau_j \in \mathsf{d}(\theta_j)$.

Definition 1.44 (Constrained switched system)**.** *Consider a switched system* SwS *with set of modes* Σ *and a timed automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$. *The system* SwS *is said to be* constrained *by* Aut *if all its switching signals are admissible for* Aut.

It is not difficult to see that constrained switched systems can be defined as hybrid systems. Also, note that constrained switched systems are more general than switched systems with absolute dwell time since the assumption that the system has absolute dwell time τ_{a} can be enforced by requiring that $\Theta \subseteq Q \times Q \times \Sigma \times [\tau_{\mathrm{a}}, \infty)$. Similarly, discrete-time switched systems can be seen as continuous-time switched systems constrained by a timed automaton with $\Theta \subseteq Q \times Q \times \Sigma \times \{1\}$.

State-dependent switching and controlled switching

The switched systems considered above assume that the switching mechanism is autonomous in the sense that the switching signal does not depend on the continuous variable nor on the input. Other classes of switched systems can be considered by adding an explicit dependence of the switching signal on the continuous variable and on the input. These systems can in general be expressed in the formalism of hybrid systems as well. In fact, the switched systems whose switching signal depends only on the continuous variable and/or on the input function can also be addressed in the setting of discontinuous control systems, but it is often easier to think about them as switched systems with state-dependent or controlled switching signal (see, e.g., Sun and Ge, 2011, Chapters 3 and 4).

In this thesis, we will mainly focus on switched systems with autonomous switching mechanisms, but we will also discuss some applications of our results for switched systems under state-dependent and controlled switching in Subsection 3.3.4 (see Example 3.6) and in Section 3.5. Further discussion on the theory and applications of switched systems with state-dependent and controlled switching can be found in Liberzon (2003) and Sun and Ge (2011).

1.3.2 Stability theory of switched linear systems

In this subsection, we restrict our attention to switched linear systems.

Definition 1.45 (Switched linear system)**.** *A switched system* $\mathsf{SwS} \sim (\mathbb{R}^n, U, \{f_i\}_{i\in\Sigma})$ *is said to be a* switched linear system *if* $U = \mathbb{R}^m$ *and for all* $i \in \Sigma$*, there is* $A_i \in \mathbb{R}^{n\times n}$ *and* $B_i \in \mathbb{R}^{n\times m}$ *such that* $f_i(x, u) = A_i x + B_i u$ *for all* $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$*.*

Switched linear systems can be more concisely described by specifying only the state space $\mathbb{R}^n$, the input space $\mathbb{R}^m$ and the matrices A_i and B_i for each $i \in \Sigma$. Thus, following the notation of the Subsection 1.3.1, we will write $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$ to mean that SwS is a switched linear system with state space $\mathbb{R}^n$, input space $\mathbb{R}^m$, set of matrices $\{A_i\}_{i\in\Sigma}$ and $\{B_i\}_{i\in\Sigma}$, and possibly additional assumptions on the switching signal (typically specified at the introduction of SwS). If SwS is autonomous, then we simply write $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$.

The generator of a switched linear system is linear in the initial state and the input function.

Proposition 1.46 (Linearity of the generator)**.** *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$*. The* generator χ *of* SwS *is linear in its third and fifth arguments, meaning that for all* $t_0, t_1 \in \mathbb{T}_{\geq 0}$*,* $t_1 \geq t_0$*,* $\sigma \in \mathcal{S}$*,* $x_1, x_2 \in \mathbb{R}^n$*,* $u_1, u_2 \in \mathcal{U}$ *and* $\mu \in \mathbb{R}$*,* $\chi(t_1, t_0, \mu x_1 + x_2, \sigma, \mu u_1 + u_2) = \mu\chi(t_1, t_0, x_1, \sigma, u_1) + \chi(t_1, t_0, x_2, \sigma, u_2)$*.*

Proof. See, e.g., Antsaklis and Michel (2006, Eq. 14.4 and 15.14 in Chapter 1). □

The linearity of the generator leads to the concept of fundamental matrix solution of autonomous switched linear systems.

Definition 1.47 (Fundamental matrix solution)**.** *Consider an autonomous switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *with generator* $\chi : \mathcal{T} \times \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}^n$*. The* fundamental matrix solution *of* SwS*, denoted by* $\mathring{\chi}(\cdot, \cdot, \cdot; \mathsf{SwS})$ *(or* $\mathring{\chi}$ *when* SwS *is clear from the context), is the function* $\mathring{\chi} : \mathcal{T} \times \mathcal{S} \to \mathbb{R}^{n\times n}$ *defined by* $\mathring{\chi}(t_1, t_0, \sigma)$ *is the unique matrix such that for all* $x \in \mathbb{R}^n$*,* $\mathring{\chi}(t_1, t_0, \sigma)x = \chi(t_1, t_0, x, \sigma)$*.*

If SwS is discrete-time, the fundamental matrix solution has the following closed-form expression: $\mathring{\chi}(t_1, t_0, \sigma) = A_{\sigma(t_1-1)} A_{\sigma(t_1-2)} \cdots A_{\sigma(t_0)}$. Similarly, if SwS is continuous-time, then it can be expressed as $\mathring{\chi}(t_1, t_0, \sigma) = e^{A_{\sigma(\tau_J)}(\tau_{J+1}-\tau_J)} e^{A_{\sigma(\tau_{J-1})}(\tau_J-\tau_{J-1})} \cdots e^{A_{\sigma(\tau_0)}(\tau_1-\tau_0)}$, where $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_J$ are the discontinuity points of σ in the interval $[t_0, t_1)$, and $\tau_0 = t_0$ and $\tau_{J+1} = t_1$.

Stability notions for switched linear systems

We refine the notion of stability for switched linear systems.

Definition 1.48 (Stability of switched linear systems)**.** *Consider an autonomous switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$.

- *We say that* SwS *is* globally asymptotically stable *(or* stable*, or* GAS*) if there is a class-*$\mathcal{KL}$ *function* β *such that for every* $x \in \mathbb{R}^n$, $\sigma \in \mathcal{S}$ *and* $t \in \mathbb{T}_{\geq 0}$, $\|\chi(t, x, \sigma)\| \leq \beta(\|x\|, t)$;
- *We say that* SwS *is* globally exponentially stable *(or* exponentially stable*, or* GES*) if it is GAS with function* β *defined by* $\beta(r, t) = Cre^{-\lambda t}$ *for all* $r, t \geq 0$ *and for some* $C \geq 0$ *and* $\lambda > 0$.

The difference with the definition of stability for hybrid systems (see Definition 1.19 in Subsection 1.1.3) is that in Definition 1.48, the uniform convergence property is required only for complete trajectories. If the system has only complete trajectories, then the two definitions coincide. This is the case for instance for switched linear systems with dwell time or average dwell time, and for discrete-time switched linear systems under arbitrary switching. For continuous-time switched linear systems under arbitrary switching, the above notion of stability coincides with the one for hybrid systems *applied on the associated differential inclusion* (see Definition 1.38), as explained in Proposition 1.51 below.

For switched linear systems, the notions of stability and exponential stability are equivalent.

Proposition 1.49. *An autonomous switched linear system is GES if and only if it is GAS.*

Proof. The proof relies on the linearity of the generator and on the shift-invariance of $\mathcal{S}$ (Proposition 1.35); see, e.g., Sun and Ge (2011, Proposition 2.13). □

Stability under arbitrary switching

The stability analysis of switched linear systems under arbitrary switching has received a great deal of attention in the literature because of its pervasiveness in theoretical and practical problems (see, e.g., Jungers, 2009). Of particular relevance for this question are the concepts of *Lyapunov exponent*, named after Lyapunov's indirect method, and its cousin concepts of *joint spectral radius*, introduced by Rota and Strang (1960), and *generalized spectral radius*, introduced by Daubechies and Lagarias (1992), which account for the maximal rate of exponential growth of the trajectories of switched linear systems under arbitrary switching.

Definition 1.50 (Joint spectral radius and generalized spectral radius). *Consider an autonomous switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *under arbitrary switching. The* Lyapunov exponent *(or* joint spectral radius[8]*) of* SwS*, denoted by* $\hat{\varrho}(\mathsf{SwS})$*, is defined by*

$$\hat{\varrho}(\mathsf{SwS}) = \lim_{T\to\infty} \frac{1}{T} \log(\sup\{\|\mathring{\chi}(T,0,\sigma)\| : \sigma \in \mathcal{S}\})$$

(the limit exists by Fekete's lemma*; see, e.g., Jungers, 2009, Lemma 1.1). The* generalized spectral radius[9] *of* SwS*, denoted by* $\check{\varrho}(\mathsf{SwS})$*, is defined by*

$$\check{\varrho}(\mathsf{SwS}) = \limsup_{T\to\infty} \frac{1}{T} \log(\sup\{\varrho(\mathring{\chi}(T,0,\sigma)) : \sigma \in \mathcal{S}\}),$$

where $\varrho(A)$ *stands for the spectral radius of* $A \in \mathbb{R}^{n\times n}$*.*

The joint spectral radius is a measure of the stability of the system in the following sense.

Proposition 1.51 (Joint spectral radius and stability). *Consider an autonomous switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *under arbitrary switching. The following are equivalent:*

1. $\hat{\varrho}(\mathsf{SwS}) < 0$*;*
2. SwS *is GES;*
3. *For all* $x \in \mathbb{R}^n$ *and* $\sigma \in \Sigma$*, it holds that* $\lim_{t\to\infty} \chi(t,x,\sigma) = 0$*.*

If SwS *is continuous-time, then any (and thus all) of the above statements is equivalent to any (and thus all) of the following statements:*

4. HySys *is GES;*
5. *Any trajectory* $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ *of* HySys *satisfies* $\lim_{t\to\infty} \phi(t) = 0$*;*

where HySys *is the differential inclusion associated to* SwS *(see Definition 1.38).*

Proof. See Appendix A.1.3.[10] □

[8]The joint spectral radius is generally defined for discrete-time switched linear systems as $e^{\hat{\varrho}(\mathsf{SwS})}$ (see, e.g., Jungers, 2009), but here for the simplicity of notation, we use the same definition for continuous-time and discrete-time systems.

[9]Same comment as for the joint spectral radius.

[10]Surprisingly, we did not find a convincing proof of this result for the case of continuous-time systems in the literature. Therefore, we present a proof of it in the appendix.

In fact, it follows from the previous proposition that the notions of joint spectral radius and generalized spectral radius coincide. This result is known as the *joint spectral radius theorem* and was first proved by Berger and Wang (1992) for discrete-time switched linear systems.

Proposition 1.52 (Joint spectral radius theorem)**.** *Consider an autonomous switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *under arbitrary switching. It holds that* $\hat{\varrho}(\mathsf{SwS}) = \check{\varrho}(\mathsf{SwS})$.

Proof. See Berger and Wang (1992) for the original proof in the case of discrete-time systems; see also Elsner (1995, Theorem 1) for a simplified proof (also in the case of discrete-time systems). The case of continuous-time systems is similar. □

Stable switched linear systems under arbitrary switching admit a Lyapunov function, sometimes called a *common Lyapunov function* since it is a Lyapunov function for each individual mode. Moreover, due to the linearity of the system, the common Lyapunov function can be chosen to be a norm in $\mathbb{R}^n$ (see, e.g., Jungers, 2009, Proposition 1.4, or Sun and Ge, 2011, Theorem 2.15). However, even though such a common Lyapunov norm is guaranteed to exist for stable switched linear systems under arbitrary switching, its computation can be challenging due to the complexity of its expression (see, e.g., Jungers, 2009, Section 2.3, and Sun and Ge, 2011, Section 2.4). Moreover, the framework of common Lyapunov functions does not allow to study the stability of switched linear systems with slow-switching conditions or with constrained switching signals. Therefore, we introduce below the theory of multiple and path-complete Lyapunov functions, addressing these issues.

1.3.3 Multiple Lyapunov functions and path-complete Lyapunov theory

Multiple Lyapunov functions were introduced by Branicky (1998) as a tool for the stability analysis of switched and hybrid systems with state-dependent or controlled switching. They were later generalized to account for switched systems under slow- and/or fast-switching assumptions[11] (see, e.g., Liberzon, 2003, Section 2.3, and Zhang and Gao, 2010).

In a nutshell, a multiple Lyapunov function is a set of Lyapunov functions, typically one or more per mode of the system, with a minimal rate of decrease

[11] Essentially, the slow-switching assumptions require that the system dwells sufficiently long in "stable" modes, and the fast-switching assumptions require that it does not dwell to long in "unstable" modes.

(which can be negative if the mode is unstable) along the trajectories of the associated mode; see Example 1.5 for an illustration. The value of the Lyapunov function depends thus on the state and on the mode of the system, and needs not necessarily decrease at the switching times. However, if the multiple Lyapunov function decreases on the long term (e.g., by imposing slow- and/or fast-switching conditions), then the system is stable.

Multiple Lyapunov functions were extensively studied, with a special focus on the computational aspects, in the context of discrete-time or constrained switched systems, under the name of *path-complete Lyapunov functions* (see, e.g., Ahmadi et al., 2014, and Philippe, 2017).

Definition 1.53 (Path-complete Lyapunov function)**.** *Consider an autonomous switched system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{f_i\}_{i\in\Sigma})$*. A* path-complete Lyapunov function *for* SwS *is an ordered pair* $(\mathsf{Aut}, \{V_q\}_{q\in Q})$ *where* $\mathsf{Aut} = (Q, \Sigma, \Theta)$ *is a timed automaton accepting every switching signal* $\sigma \in \mathcal{S}$*, and* $\{V_q\}_{q\in Q}$ *is a set of functions from* $\mathbb{R}^n$ *to* $\mathbb{R}_{\geq 0}$ *that are* $\mathcal{K}$*-equivalent to* $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$*, such that there is a class-*$\mathcal{K}$ *function* α *satisfying that for every transition* $\theta \in \Theta$*,* $x \in \mathbb{R}^n$ *and* $\tau \in \mathsf{d}(\theta)$*,* $V_{\mathsf{t}(\theta)}(\chi(\tau, x; f_{\mathsf{i}(\theta)})) - V_{\mathsf{s}(\theta)}(x) \leq -\alpha(V_{\mathsf{s}(\theta)}(x))$*, where* $\chi(\cdot,\cdot; f_i)$ *is the generator of the dynamical system associated to the mode* i*.*

Example 1.5. Consider a continuous-time switched linear system $\mathsf{SwS} \sim (\mathbb{R}^1, \{f_i\}_{i\in\Sigma})$ with two modes: $f_1(x) = -x$ and $f_2(x) = 2x$; and assume that SwS has dwell time 3 (units of time) for mode 1 and cannot stay longer than 1 (unit of time) in mode 2. Then, the timed automaton and the quadratic functions depicted in Figure 1.13 provide a path-complete Lyapunov function for SwS. Indeed, for the transition $(\mathsf{a}, \mathsf{b}, 2, [0, 1])$, it holds that for any $x \in \mathbb{R}$ and $\tau \in [0, 1]$, $|\chi(\tau, x; f_2)| \leq e^2|x|$, so that $V_{\mathsf{b}}(\chi(\tau, x; f_2)) \leq e^4x^2 < 60x^2 = V_{\mathsf{a}}(x)$. As for the transition $(\mathsf{b}, \mathsf{a}, 2, [3, \infty))$, it holds that for any $x \in \mathbb{R}$ and $\tau \in [3, \infty)$, $|\chi(\tau, x; f_1)| \leq e^{-3}|x|$, so that $V_{\mathsf{a}}(\chi(\tau, x; f_1)) \leq 60e^{-6}x^2 < x^2 = V_{\mathsf{b}}(x)$.

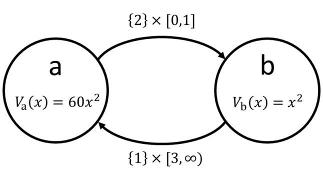

Figure 1.13: A path-complete Lyapunov function for SwS in Example 1.5, consisting in a timed automaton (same as in Figure 1.12) together with quadratic functions associated to each state of the automaton.

The existence of a path-complete Lyapunov function ensures stability of the system.

Proposition 1.54 (Stability from path-complete Lyapunov function)**.** *Consider a switched system* SwS *and assume that there is a path-complete Lyapunov* *function for* SwS*. Then,* SwS *is stable with respect to the cost* $\mathfrak{C} : \mathbb{R}^n \times \Sigma \to \mathbb{R}$ *defined by* $\mathfrak{C}(x, \sigma) = \|x\|$*.*

Proof. See, e.g., Ahmadi et al. (2014, Theorem 2.4) for the case of discrete-time switched linear systems. The general case is similar. □

The framework of multiple Lyapunov functions can also be used to study the stability of switched systems with average dwell time.

Proposition 1.55. *Consider an autonomous switched system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{f_i\}_{i\in\Sigma})$*. Let* $\mathsf{Aut} = (Q, \Sigma, \Theta)$ *be a timed automaton accepting any right-continuous, piecewise constant function from* $\mathbb{T}_{\geq 0}$ *to* Σ*, and* $\{V_q\}_{q\in Q}$ *be a set of functions from* $\mathbb{R}^n$ *to* $\mathbb{R}_{\geq 0}$ *that are* $\mathcal{K}$*-equivalent to* $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$*. Assume that there is* $\lambda > 0$ *such that for all* $\theta \in \Theta$*,* $V_{\mathsf{s}(\theta)}(\chi(t, x; A_{\mathsf{i}(\theta)})) \leq e^{-\lambda t} V_{\mathsf{s}(\theta)}(x)$ *for all* $x \in \mathbb{R}^n$ *and* $t \in \mathbb{T}_{\geq 0}$*, and there is* $\mu > 0$ *such that for all* $\theta \in \Theta$*,* $V_{\mathsf{s}(\theta)}(x) \leq \mu V_{\mathsf{t}(\theta)}(x)$ *for all* $x \in \mathbb{R}^n$*. Also, assume that* SwS *has average dwell time* τ_{a} *with* $\tau_{\mathrm{a}} > \log(\mu)/\lambda$*. Then,* SwS *is stable with respect to the cost* $\mathfrak{C} : \mathbb{R}^n \times \Sigma \to \mathbb{R}$ *defined by* $\mathfrak{C}(x, \sigma) = \|x\|$*.*

Proof. See, e.g., Liberzon (2003, Theorem 3.2) for the case of continuous-time systems. The case of discrete-time systems is similar. □

In particular, it follows that any switched linear system with stable individual modes is stable if it has a large enough average dwell time (see, e.g., Liberzon, 2003, Subsection 3.2.2).

1.4 Abstractions of dynamical systems

The approach of abstraction consists in representing a dynamical or hybrid system by a finite system, called a *symbolic model*. The analysis and control of the original system can then be achieved from the study of the symbolic model, provided that the latter provides a sufficiently accurate representation of the original system (a property formalized thanks to the notion of simulation relation). The algorithmic theory of abstraction generally comprises two aspects: the construction of the symbolic model, and the analysis of the symbolic model, typically using techniques from graph and automata theory (the frontier between the two aspects can sometimes be fuzzy, for instance in the case of adaptive algorithms).

In this section, we explain how to compute accurate abstractions of smooth dynamical systems and how to use the resulting symbolic model to deduce several properties of the original system, namely regarding the location of its invariant sets. These techniques will be used in Section 2.3 for the analysis of

p-dominance and hyperbolicity of nonlinear smooth dynamical systems. Let us mention that a common limitation of the techniques of abstraction are their poor scalability with respect to the dimension of the system (called the *curse of dimensionality*); and although no complexity analysis is provided, the techniques presented in this section are no exception to it.

The section is organized as follows. In Subsection 1.4.1, we define the concepts of symbolic model and simulation relation. In Subsection 1.4.2, we describe an algorithm to construct accurate symbolic models based on uniform grid discretization of the state space and the sensitivity matrix of the system. Finally, in Subsection 1.4.3, we explain how to locate the invariant sets of dynamical systems based on their symbolic models.

References. The techniques of abstraction and formal verification for the analysis and design of dynamical and hybrid systems have received a lot of attention from the control community in recent years, with great theoretical and practical advances; see, e.g., Osipenko (2007), Tabuada (2009), Lee and Seshia (2017) and Mitra (2021) for introductions. Some of these techniques are implemented in state-of-the-art software tools, such as Pessoa, CoSyMA or SCOTS, for the abstraction and symbolic control, and CAPD, Flow* or JuliaReach, for the reachability analysis. Our main references for this section are Osipenko (2007) for the definition of symbolic models and their use to locate the invariant sets of dynamical systems, and Tabuada (2009) for the definition of transition systems and simulation relations.

Notation. In this section, all considered hybrid systems are smooth dynamical systems, and thus for the sake of brievety, we will refer to them simply as *dynamical systems*. Also, we will consider both continuous-time systems and discrete-time systems, and we will use the symbol $\mathbb{T}$ to denote the time domain of the system, as it should be clear from the context whether $\mathbb{T} = \mathbb{R}$ (continuous-time systems) or $\mathbb{T} = \mathbb{Z}$ (discrete-time systems). The Minkowski sum of $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$ (or $\{x\} \subseteq \mathbb{R}^n$) is denoted by $A + B$ (or $A + x$). If $A \subseteq \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$, then MA is the image of A by M, i.e., $MA = \{Mx : x \in A\}$. Finally, the Hadamard (or component-wise) product of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ is denoted by $x \odot y$.

1.4.1 Symbolic models and simulation relations

We introduce below the definition of a symbolic model of a dynamical system. First, we remind the definition of a directed graph.

Definition 1.56 (Directed graph)**.** *A* directed graph *is an ordered pair* (Q, E) *where* Q *is a finite set called the* set of nodes*, and* $E \subseteq Q \times Q$ *is called the* set of edges.

A symbolic model can be seen as a directed graph, for which each node of the graph is associated to a region of the state space and the edges of the graph represent the possible transitions of the system from one region to another.

Definition 1.57 (Symbolic model)**.** *Consider a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$ *and let* $T \in \mathbb{T}_{>0}$ *be a time step. A* symbolic model *of* Sys *with time step* T *is an ordered pair* $(\{O_q\}_{q\in Q}, E)$*, where* Q *is a finite set,* $O_q \subseteq \mathbb{R}^n$ *for each* $q \in Q$*, and* $E \subseteq Q \times Q$ *is a set of* edges *satisfying that for every* $q_1, q_2 \in Q$*,* $\{\chi(T, x) : x \in O_{q_1}\} \cap O_{q_2} \neq \emptyset$ *implies that* $(q_1, q_2) \in E$*.*

See Figure 1.14 for an illustration.

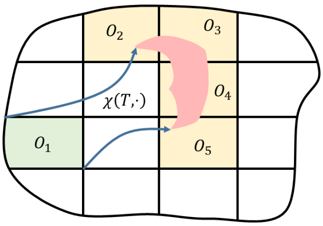

Figure 1.14: Symbolic model of a dynamical system. The image of the region O_1 is depicted in red. Since the image of O_1 intersects the regions O_2, O_3, O_4 and O_5, the set of edges of the symbolic model contains at least the edges $(1, 2)$, $(1, 3)$, $(1, 4)$ and $(1, 5)$.

The notion of symbolic model closely connects with the one of simulation relation between two transition systems.

Definition 1.58 (Transition system)**.** *A* transition system *is an ordered pair* (X, G) *where* X *is a set, called the* state space*, and* $G : X \rightrightarrows X$ *is a set-valued map, called the* transition map*.*

Definition 1.59 (Simulation relation)**.** *Consider two transition systems* $\mathsf{Trans} = (X, G)$ *and* $\mathsf{Trans}' = (X', G')$*. A* simulation relation *from* Trans *to* Trans' *(meaning that* Trans' simulates Trans*) is a set-valued map* $R : X \rightrightarrows X'$ *such that* $\operatorname{dom} R = X$[12] *and for every* $x_1 \in X$*,* $x_2 \in G(x_1)$ *and* $x_1' \in R(x_1)$*, there is* $x_2' \in R(x_2)$ *such that* $x_2' \in G'(x_1')$*.*

The connection between transition systems, simulation relations and symbolic models is as follows. Given a dynamical system $\mathsf{Sys} = (\mathbb{R}^n, f)$, a set $\Lambda \subseteq \mathbb{R}^n$ forward invariant for Sys and a time step $T \in \mathbb{T}_{>0}$, we define the associated transition system by $\mathsf{Trans} = (\Lambda, G)$ where $G : \Lambda \rightrightarrows \Lambda$ is defined by $G(x) = \{\chi(T, x)\}$. Similarly, a directed graph (Q, E) can be seen as the transition system $\mathsf{Trans}' = (Q, G')$ where $G' : Q \rightrightarrows Q$ is defined by $G'(q_1) = \{q_2 \in Q : (q_1, q_2) \in E\}$.

Proposition 1.60. *Let* Sys*,* Λ*,* T *and* Trans *be as above. Let* $(\{O_q\}_{q\in Q}, E)$ *be a symbolic model for* Sys *with time step* T *and assume that* $\Lambda \subseteq \bigcup_{q\in Q} O_q$*.*

[12] As a reminder, this means that $R(x) \neq \emptyset$ for all $x \in X$.

Let Trans' *be as above. Then, the set-valued map* $R : \Lambda \rightrightarrows Q$*, defined by* $R(x) = \{q \in Q : x \in O_q\}$*, is a simulation relation from* Trans *to* Trans'*.*

Proof. Straightforward from the definitions of symbolic models and simulation relations. □

1.4.2 Construction of symbolic models

The construction of symbolic models for a given dynamical system can be approached in a two-step fashion, namely by (i) defining the regions O_q of the state space for each $q \in Q$, and (ii) computing the set of edges E capturing the transitions of the system between the different regions.

For the step (i), a popular approach (mainly due to the simplicity of its implementation) consists in defining the regions O_q as rectangular cells aligned according to a uniform grid. This is the approach that we follow in this work.

To describe this approach, we first introduce the notion of hyper-rectangle.

Definition 1.61 (Hyper-rectangle)**.** *Let* $x_1, x_2 \in \mathbb{R}^n$*. The* hyper-rectangle *spanned by* x_1 *and* x_2*, denoted as* $[x_1, x_2]$*, is defined by* $[x_1, x_2] = \{x \in \mathbb{R}^n : x - x_1 \in (\mathbb{R}_{\geq 0})^n,\ x_2 - x \in (\mathbb{R}_{\geq 0})^n\}$*.*

Definition 1.62 (Uniform grid discretization)**.** *Given a bounded subset* $\Omega \subseteq \mathbb{R}^n$ *and a discretization step* $h \in (\mathbb{R}_{>0})^n$*, we define the* uniform grid discretization *of* Ω *with step* h*, denoted by* $\mathrm{Grid}(\Omega, h)$*, as the set of all subsets* $O \subseteq \mathbb{R}^n$ *of the form* $O = h \odot z + [-\frac{1}{2}h, \frac{1}{2}h]$ *where* $z \in \mathbb{Z}^n$ *and* $O \cap \Omega \neq \emptyset$*.*

Since Ω is bounded, it holds that $\mathrm{Grid}(\Omega, h)$ in Definition 1.62 is finite and thus we can index its elements as $\mathrm{Grid}(\Omega, h) = \{O_q\}_{q \in Q}$ where $Q = \{1, \ldots, N\}$.

The step (ii) in the process of constructing the symbolic model then amounts to compute a set of edges that captures the transitions of the system between the regions $\{O_q\}_{q \in Q}$. This can be done for instance by computing, for each $q \in Q$, an over-approximation Y_q of the set $\{\chi(T, x) : x \in O_q\}$, where $T \in \mathbb{T}_{>0}$ is the time step and χ is the generator of the system, and letting E be the set of ordered pairs $(q_1, q_2) \subseteq Q \times Q$ such that $Y_{q_1} \cap O_{q_2} \neq \emptyset$. Various approaches have been proposed in the literature to compute the over-approximations $\{Y_q\}_{q \in Q}$; see, e.g., Reißig (2011) (polyhedral over-approximations), Chen et al. (2013) (*Taylor model flowpipes*), Reißig et al. (2017) (hyper-rectangle over-approximations using a *growth bound function*), and Gruenbacher et al. (2020) (ellipsoidal over-approximations using interval matrices).

In this thesis, we use a technique based on the *sensitivity matrix* $\frac{\partial \chi}{\partial x}$ to compute the over-approximations $\{Y_q\}_{q \in Q}$.[13] The underlying idea is to use

[13] See also Proposition 1.30 in Subsection 1.2.3 for the properties of the sensitivity matrix.

the sensitivity matrix to compute, for each $q \in Q$, a first-order approximation of the set $\{\chi(T, x) : x \in O_q\}$ and then inflate the first-order approximation to get an over-approximation. This technique connects for instance with the one in Gruenbacher et al. (2020); the difference is that we rely on continuity assumptions on the sensitivity matrix, instead of interval matrices, to inflate the first-order approximation.

Our technique for the computation of the over-approximations works as follows. Let $\{O_q\}_{q\in Q}$ be the regions obtained from a uniform discretization grid $\mathrm{Grid}(\Omega, h)$. For the simplicity of notation, we assume that h is isotropic, meaning that $h = [r, \ldots, r]^\top$ for some $r > 0$, so that the regions $\{O_q\}_{q\in Q}$ are hyper-cubic. This assumption can be made without loss of generality by using a rescaling of the axis if necessary.

Let $T \in \mathbb{T}_{>0}$ be the time step and χ be the generator of the system. It is assumed that $\frac{\partial\chi}{\partial x}(T, \cdot)$ is uniformly Lipschitz continuous with respect to the matrix and vector ∞-norms.[14]

Assumption 1.63. *There is $L \geq 0$ such that for every $x_1, x_2 \in \mathbb{R}^n$, $\|\frac{\partial\chi}{\partial x}(T, x_1) - \frac{\partial\chi}{\partial x}(T, x_2)\|_\infty \leq L\|x_1 - x_2\|_\infty$, where $\|\cdot\|_\infty$ stands for the matrix and vector ∞-norms.*

An over-approximation of the reachable set from any hyper-cubic region can then be obtained as follows.

Proposition 1.64 (Over-approximation of reachable sets from hyper-cubes)**.** *Consider a dynamical system $\mathsf{Sys} = (\mathbb{R}^n, f)$ and a time step $T \in \mathbb{T}_{>0}$. Let Assumption 1.63 hold with Lipschitz constant $L \geq 0$. Then, for any $x_0 \in \mathbb{R}^n$ and $r_0 \geq 0$, it holds that*

$$\{\chi(T, x) : x \in x_0 + [-r_0, r_0]^n\} \subseteq \chi(T, x_0) + \frac{\partial\chi}{\partial x}(T, x_0)[-r_0, r_0]^n + \frac{L}{2}[-r_0^2, r_0^2]^n.$$

Proof. See Appendix A.1.4. □

See Figure 1.15 for an illustration.

It follows from Proposition 1.64 the following sufficient condition to ensure that the reachable set from a given hyper-cubic region does not intersect another given hyper-cubic region.

The motivation to use the sensitivity matrix to compute the over-approximations is that this matrix will also be used in other analysis techniques based on the computed symbolic model, namely, for the study of p-dominance and hyperbolicity of nonlinear dynamical systems (see Section 2.3).

[14] For the purpose of computing symbolic models, it is sufficient to assume that $\frac{\partial\chi}{\partial x}(T, \cdot)$ is uniformly Lipschitz on $\bigcup \mathrm{Grid}(\Omega, h)$, but here, for the simplicity of notation, we assume that it is uniformly Lipschitz on $\mathbb{R}^n$.

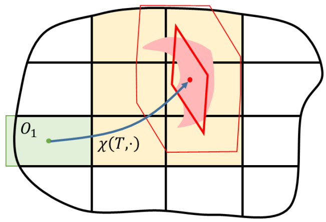

Figure 1.15: Over-approximation of the image of the region O_1 using the technique of Proposition 1.64. The image of O_1 is depicted in light red (see also Figure 1.14). The quadrilateral contour in bold red is the first-order approximation of the image of O_1, obtained by computing the linear image of the rectangle O_1 by the sensitivity matrix of the system at the center of O_1. The surrounding polygonal contour in thin red is the inflation of the first-order approximation using the Lipschitz constant of the system, providing an over-approximation of the image of O_1.

Corollary 1.65. *Consider a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$ *and a time step* $T \in \mathbb{T}_{>0}$. *Let Assumption 1.63 hold with Lipschitz constant* $L \geq 0$. *Then, for any* $x_0, x_1 \in \mathbb{R}^n$ *and* $r_0, r_1 \geq 0$, *it holds that* $\{\chi(T, x) : x \in x_0 + [-r_0, r_0]^n\} \cap (x_1 + [-r_1, r_1]^n) = \emptyset$ *if*

$$x_1 - \chi(T, x_0) \notin \frac{\partial \chi}{\partial x}(T, x_0)[-r_0, r_0]^n + \Big[- r_1 - \frac{L}{2} r_0^2, r_1 + \frac{L}{2} r_0^2 \Big]^n. \tag{1.3}$$

Proof. See Appendix A.1.5. □

The membership problem (1.3) can be decided by computing a representation of the right-hand side of (1.3) as a system of linear inequalities (called a *half-space representation* or *H-representation*; see, e.g., Legat, 2020, Subsection 1.3.1), and verifying that the left-hand side of (1.3) satisfies these linear inequalities. For low-dimensional systems, computing a half-space representation of the right-hand side can be achieved in reasonable time. However, in higher dimensions, it can be advantageous to compute the half-space representation of an over-approximation of right-hand side to reduce the computation time; the price to pay being to increase the conservatism of the criterion for $\{\chi(T, x) : x \in x_0 + [-r_0, r_0]^n\} \cap (x_1 + [-r_1, r_1]^n) = \emptyset$. This is the approach that we used in our implementation of methods to compute abstractions of nonlinear dynamical systems[15]; the details are omitted here since the computation of symbolic models is not the main objective of this thesis.

[15] See `https://github.com/guberger/Dominance.jl/blob/main/src/symbolic_model_from_system.jl`.

1.4.3 System analysis from symbolic models

Symbolic models can be used to address a wide range of problems in systems and control theory. They can be used for instance for the stability or safety analysis of dynamical systems, and for the controller design of dynamical systems with input (see, e.g., Tabuada, 2009). In this thesis, we will use symbolic models for the study of p-dominance and hyperbolicity of nonlinear dynamical systems (see Section 2.3). This will require among others to locate the maximal invariant set of such systems, which can be done as explained below.

To do this, we first introduce the notion of essential graphs, which are directed graphs in which each node has an incoming edge and an outgoing edge.

Definition 1.66 (Essential graph)**.** *A directed graph (Q, E) is said to be* essential *if for every $q \in Q$, there is $q^+ \in Q$ such that $(q, q^+) \in E$ and there is $q^- \in Q$ such that $(q^-, q) \in E$.*

It is easily seen that the union of two essential graphs is essential. Hence, given a directed graph Gp, we define the *maximal essential subgraph* of Gp as the union of all essential subgraphs of Gp.

This leads to the following characterization of the maximal invariant sets of dynamical systems based on their symbolic models; see also Figure 1.16 for an illustration.

Proposition 1.67 (Symbolic models and maximal invariant sets)**.** *Consider an invertible dynamical system $\mathsf{Sys} = (\mathbb{R}^n, f)$, a subset $\Omega \subseteq \mathbb{R}^n$ and time step $T \in \mathbb{T}_{>0}$. Let $(\{O_q\}_{q\in Q}, E)$ be a symbolic model of Sys with time step T and with $\Omega = \bigcup_{q\in Q} O_q$. Let Λ be the maximal invariant set of Sys included in $\{\chi(t, x) : t \in [0, T) \cap \mathbb{T},\, x \in \Omega\}$. Let (Q', E') be the maximal essential subgraph of (Q, E). Then, it holds that $\Lambda \subseteq \{\chi(t, x) : t \in [0, T) \cap \mathbb{T},\, x \in O_q,\, q \in Q'\}$.*

Proof. See, e.g., Osipenko (2007, Theorem 44) for the case of discrete-time systems and with time step $T = 1$. The case of continuous-time systems and $T \neq 1$ is similar. □

Moreover, under mild assumptions on the construction of the symbolic models (which are satisfied for instance by the construction given in Subsection 1.4.2), the set $\{\chi(t, x) : t \in [0, T) \cap \mathbb{T},\, x \in O_q,\, q \in Q'\}$ in Proposition 1.67 converges to A when the diameter of the regions $\{O_q\}_{q\in Q}$ of the symbolic model tends to zero (see, e.g., Osipenko, 2007, Theorems 42 and 44).

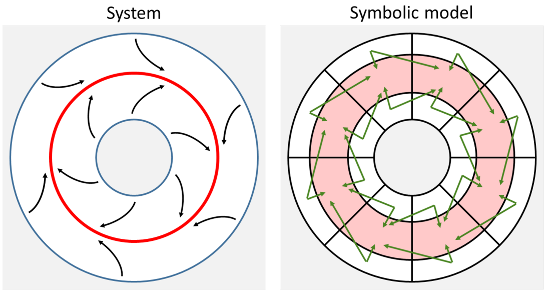

Figure 1.16: *Left:* Dynamical system whose trajectories converge from outside towards the inside of a ring and have a clockwise circular motion. The system has a maximal invariant set represented by the red line. *Right:* Symbolic model of the dynamical system using curved rectangular regions. The external (inner and outer) regions have edges (in green) only towards the regions in the middle strip (in light red). The middle strip is thus the maximal essential subgraph of the symbolic model, and it contains the maximal invariant set of the system.

1.5 Networked systems

Networked systems are systems, in which the different agents (plants, sensors, actuators, controllers, etc.) are spatially distributed and communicate through a shared, band-limited, digital communication network. The design and control of networked systems entails dealing with the non-idealities of the communication channel, such as noise, limited bandwidth, random delays, packet dropouts, etc. This can pose many challenges for the control theorist and motivated the development of a new paradigm in systems theory, where control and communication issues are integrated.

In this thesis, we will focus on the challenges posed by the quantization and the limited information flow imposed by the digital communication network. Indeed, due to the digital nature of the network, all data must be quantized before transmission, resulting in quantization error that can have large negative effects on the observability or controllability of the system. Furthermore, in applications, the capacity of the network is often limited by cost, power, physical and/or security constraints. Consequently, a major challenge in the design of such systems is to determine the minimal communication data rate that is needed to achieve a given control objective.

The quantization process is achieved by means of "devices" called *coders–decoders*, which are modeled as hybrid systems. The data rate of a coder–decoder is defined as the averaged number of bits transmitted per unit of time from the coder to the decoder. Furthermore, the question of data rate requirement for networked systems also connects with the notion of *topological entropy* of dynamical systems. This quantity, introduced in the late 60's and now ubiquitous in dynamical systems theory, measures the growth rate of the smallest number of functions necessary to approximate the trajectories of the system with arbitrary finite accuracy on bounded time intervals (the growth rate is with respect to the length of the time interval). We introduce this notion in the context of hybrid systems and we explain its connection with the minimal data rate for state estimation of these systems.

The section is organized as follows. In Subsection 1.5.1, we introduce the concepts of coder–decoder and minimal data rate for state estimation and stabilization of hybrid systems. In Subsection 1.5.2, we introduce the notion of topological entropy for hybrid systems, and we explain its connection with the minimal data rate for state estimation of hybrid systems.

References. The study of networked systems and control problems under quantization and data-rate constraints has attracted a lot of attention in recent years; see, e.g., Hespanha et al. (2007), Nair et al. (2007), Matveev and Savkin (2009) and Kawan (2013) for introductions. Our main reference for this section is Matveev and Savkin (2009). Let us also mention that several variants of the concept of topological entropy have been proposed in the literature to address further aspects of networked systems; see, e.g., Liberzon and Mitra (2018) (exponentially decreasing estimation error), Colonius et al. (2013) (feedback invariance and feedback stabilization), Colonius (2012) (exponential stabilization); Hagihara and Nair (2013) (systems with output), etc.; but will not be discussed in this thesis.

Notation. The restriction of a function $f : A \to B$ to a set $A' \subseteq A$ is denoted by $f|_{A'}$. For the sake of simplicity of notation, we will assume that all maximal trajectories (referred to as *trajectories*) of the considered hybrid systems are complete.

1.5.1 Coder, decoder and minimal data rate

We introduce below the concept of coder–decoder for the state estimation and stabilization of hybrid systems. The coder is a device that, given the past observation of the system, sends a symbol to the decoder. Based on the past received symbols, the decoder outputs an estimation of the current state of the system, or generates an input to stabilize the system. Some information about the state of the system are also considered as *universal knowledge* (meaning that it is known by the decoder without this information being sent by the coder). This information is delivered by a *universal output map*, which is a set-valued function $\mathfrak{H} : X \rightrightarrows Y$ where Y is the *universal output set* (if there is no universal information, then the universal output set is a singleton).[16]

In this thesis, we focus on coders–decoders that communicate at periodic time instants. To describe them, we introduce the concept of hybrid systems that have a periodic "sample and hold" output with respect to some output function.

Definition 1.68 (Sample-and-hold output)**.** *Consider a hybrid system* $\mathsf{HySys} = (X, X_0, U, F, G)$*, a function* $\Upsilon : X \to Y$ *and a period* $T \in \mathbb{R}_{>0}$*. We say that* HySys *has* T-sample-and-hold output *with respect to* Υ *if every trajectory* $\phi : \mathbb{R} \to X$ *of* HySys *satisfies that for all* $k \in \mathbb{N}$ *and* $t \in [kT, (k+1)T)$*,* $\Upsilon(\phi(t)) = \Upsilon(\phi(kT))$*.*

In other words, HySys has T-sample-and-hold output with respect to Υ if any trajectory of HySys is mapped by Υ to a function that is piecewise constant and changes only at periodic time instants (namely at multiples of T).

Using the above, we define the concept of coder–decoder. The coder and the decoder are in fact themselves represented as hybrid systems.

Definition 1.69 (Coder–decoder)**.** *Consider a hybrid system* $\mathsf{HySys} = (X, X_0, U, F, G)$ *and a universal output map* $\mathfrak{H} : X \rightrightarrows Y$*. A* coder–decoder *for* HySys *with universal output map* $\mathfrak{H}$ *(or just "for* HySys*" if* $\mathfrak{H}$ *is clear from the context) is a quintuple* $(T_{\mathrm{t}}, \mathsf{HySys}_{\mathrm{c}}, \Upsilon, \mathsf{HySys}_{\mathrm{d}}, K)$ *where*

- $T_{\mathrm{t}} \in \mathbb{R}_{>0}$ *is the* transmission period*;*
- $\mathsf{HySys}_{\mathrm{c}} = (X_{\mathrm{c}}, X_{\mathrm{c}0}, X, F_{\mathrm{c}}, G_{\mathrm{c}})$ *is a hybrid system, called the* coder*;*
- $\Upsilon : X_{\mathrm{c}} \to Y_{\mathrm{t}}$ *is a function, called the* transmission map*, mapping the state of the coder to a symbol from the* symbol set Y_{t}*, and* $\mathsf{HySys}_{\mathrm{c}}$ has T_{t}-sample-and-hold output *with respect to* Υ*;*
- $\mathsf{HySys}_{\mathrm{d}} = (X_{\mathrm{d}}, X_{\mathrm{d}0}, Y_{\mathrm{t}} \times Y, F_{\mathrm{d}}, G_{\mathrm{d}})$ *is a hybrid system, called the* decoder*;*
- $K : X_{\mathrm{d}} \rightrightarrows X \times U$ *is a set-valued function, called the* observer–actuator map*, mapping the state of the decoder to a subset of the state space and input space of* HySys*.*

If T_{t} *is clear from the context, we describe the coder–decoder simply with* $(\mathsf{HySys}_{\mathrm{c}}, \Upsilon, \mathsf{HySys}_{\mathrm{d}}, K)$*.*

The above definition deserves the following explanations. The coder $\mathsf{HySys}_{\mathrm{c}}$ takes as input the current state of HySys (we say that it *observes* or *measures* the

[16] The letter "$\mathfrak{H}$" is a calligraphic "H", since "H" is generally used to denote the output map of a system.

state of HySys), and based on this observation, it updates its own current state. The transmission map Υ (which can be seen as being part of the coder) outputs a symbol depending on the current state of $\mathsf{HySys}_{\mathrm{c}}$. It is assumed that the value of the symbol changes only at periodic time instants (this generally requires that some variable of $\mathsf{HySys}_{\mathrm{c}}$ behaves as a sample-and-hold). The decoder $\mathsf{HySys}_{\mathrm{d}}$ takes as inputs the symbol sent by the coder via the transmission map and the observation of the current state of HySys via the universal output map $\mathfrak{H}$. Based on these inputs, it updates its own current state. Finally, the observer–actuator map K (which can be seen as being part of the decoder) outputs a pair $(\hat{x}, u)$, where $\hat{x}$ is meant to be an estimation of the current state of HySys (in the case of a “state estimation” problem) and u is meant to be a control input for HySys (in the case of a “stabilization” problem).

We use coders–decoders for the observation and control of hybrid-systems over digital communication channels. This raises the following two questions: How can we use a coder–decoder to observe or control a hybrid system? And which requirements on the communication channel do we need for the coder–decoder to work properly? The first question leads to the notion of feedback composition of a hybrid system with a coder–decoder, introduced below and which extends the notion of feedback composition of a hybrid system and a controller (see Definition 1.18 in Subsection 1.1.3); see also Figure 1.17 for an illustration. The second question leads to the notion of communication data rate of a coder–decoder, introduced afterwards.

Definition 1.70 (Feedback composition with a coder–decoder)**.** *Consider a hybrid system* $\mathsf{HySys} = (X, X_0, U, F, G)$ *and a coder–decoder* $\mathsf{CoDec} = (\mathsf{HySys}_{\mathrm{c}}, \Upsilon, \mathsf{HySys}_{\mathrm{d}}, K)$*,* $\mathsf{HySys}_{\mathrm{c}} = (X_{\mathrm{c}}, X_{\mathrm{c}0}, X, F_{\mathrm{c}}, G_{\mathrm{c}})$*,* $\mathsf{HySys}_{\mathrm{d}} = (X_{\mathrm{d}}, X_{\mathrm{d}0}, Y_{\mathrm{t}} \times Y, F_{\mathrm{d}}, G_{\mathrm{d}})$*, for* HySys *with universal output map* $\mathfrak{H} : X \rightrightarrows Y$*. The* feedback composition *of* HySys *and* CoDec *(via* $\mathfrak{H}$*), denoted by* $\mathsf{HySys}\|_{\mathfrak{H}}\mathsf{CoDec}$ *(or* $\mathsf{HySys}\|\mathsf{CoDec}$ *if* $\mathfrak{H}$ *is clear from the context), is the autonomous hybrid system* $(X_{\mathrm{f}}, X_{\mathrm{f}0}, \{0\}, F_{\mathrm{f}}, G_{\mathrm{f}})$ *where* $X_{\mathrm{f}} = X \times X_{\mathrm{c}} \times X_{\mathrm{d}}$*,* $X_{\mathrm{f}0} = X_0 \times X_{\mathrm{c}0} \times X_{\mathrm{d}0}$*, and for* $\Box \in \{F, G\}$*,* $\Box_{\mathrm{f}}(x, x_{\mathrm{c}}, x_{\mathrm{d}}, 0) = \bigcup_{(\hat{x},u)\in K(x_{\mathrm{d}})} \Box(x, u) \times \Box_{\mathrm{c}}(x_{\mathrm{c}}, x) \times \bigcup_{y\in\mathfrak{H}(x)} \Box_{\mathrm{d}}(x_{\mathrm{d}}, \Upsilon(x_{\mathrm{c}}), y)$ *for all* $(x, x_{\mathrm{c}}, x_{\mathrm{d}}) \in X \times X_{\mathrm{c}} \times X_{\mathrm{d}}$*.*

Thus, in the context of a feedback composition, the pair $(\mathsf{HySys}_{\mathrm{c}}, \Upsilon)$ can be seen as a sequence of functions $(\Psi^{\mathrm{c}}_k)_{k\in\mathbb{N}}$ that outputs at each time $t = kT_{\mathrm{t}}$, with $k \in \mathbb{N}$, a symbol $e(k) \in Y_{\mathrm{t}}$ based on the past observation of the system:

$$e(k) \in \Psi^{\mathrm{c}}_k(\phi|_{(-\infty,kT_{\mathrm{t}}]}), \tag{1.4}$$

where $\Psi^{\mathrm{c}}_k : X^{(-\infty,kT_{\mathrm{t}}]} \rightrightarrows Y_{\mathrm{t}}$ is the *coder function* at step $k \in \mathbb{N}$ and $\phi : \mathbb{R} \to X$ is the trajectory of HySys. Similarly, the pair $(\mathsf{HySys}_{\mathrm{d}}, K)$ can be seen as a

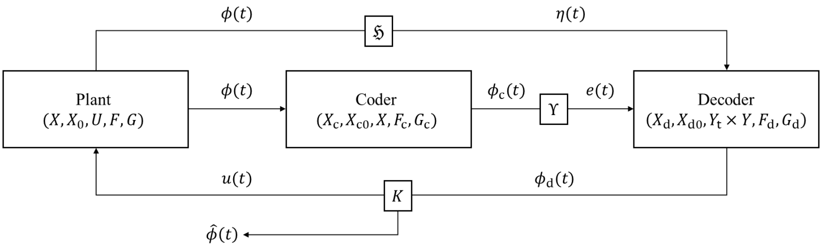

Figure 1.17: Feedback composition of a hybrid system $\mathsf{HySys} = (X, X_0, U, F, G)$ and a coder–decoder $\mathsf{CoDec} = (\mathsf{HySys}_c, \Upsilon, \mathsf{HySys}_d, K)$, $\mathsf{HySys}_c = (X_c, X_{c0}, X, F_c, G_c)$, $\mathsf{HySys}_d = (X_d, X_{d0}, Y_t \times Y, F_d, G_d)$, for HySys with universal output map $\mathfrak{H} : X \rightrightarrows Y$.

sequence of functions $(\Psi^d_k)_{k\in\mathbb{N}}$ that, given the past received symbols and the past observation of the universal output of the system, generates an estimate $\hat{\phi}(t) \in X$ of the state of HySys and an input $u(t) \in U$ for HySys:

$$(\hat{\phi}(t), u(t)) \in \Psi^d_k(e(0), \ldots, e(k), \eta|_{(-\infty,t]}), \qquad \text{for all } t \in [kT_t, (k+1)T_t), \tag{1.5}$$

where $\Psi^d_k : (Y_t)^{k+1} \times \bigcup_{t\in[kT_t,(k+1)T_t)} Y_t^{(-\infty,t]} \rightrightarrows X \times U$ is the *decoder function* at step $k \in \mathbb{N}$ and $\eta : \mathbb{R} \to Y$ is an observation of ϕ via $\mathfrak{H}$, i.e., $\eta(t) \in \mathfrak{H}(\phi(t))$ for all $t \in \mathbb{R}$.

Remark 1.10. The description of the coder–decoder via a coder and a decoder function, as in (1.4)–(1.5), is useful for its conciseness. However, this description is weaker than the one in Definition 1.69 since it does not require that Ψ_c and Ψ_d can be implemented as hybrid systems. Therefore, when we describe a coder–decoder via coder and decoder functions, it is important to keep in mind that these functions must be *implementable as hybrid systems.*

The feedback composition of a hybrid system with a coder–decoder can be used to estimate the state of the system or to stabilize it.

Definition 1.71 (State estimation from a coder–decoder)**.** *Consider an autonomous hybrid system* $\mathsf{HySys} = (X, X_0, U, F, G)$ *and a cost function* $\mathfrak{C} : X \times X \to \mathbb{R}_{\geq 0}$*. Let* $\mathsf{CoDec} = (\mathsf{HySys}_c, \Upsilon, \mathsf{HySys}_d, K)$*,* $\mathsf{HySys}_c = (X_c, X_{c0}, X, F_c, G_c)$*,* $\mathsf{HySys}_d = (X_d, X_{d0}, Y_t \times Y, F_d, G_d)$*, be a coder–decoder for* HySys *with universal output map* $\mathfrak{H} : X \rightrightarrows Y$*. Let* $\epsilon > 0$*. We say that* CoDec observes HySys *with accuracy* ϵ *(or* ϵ-observes HySys*) with respect to* $\mathfrak{C}$ *(via* $\mathfrak{H}$*) if the feedback composition* $\mathsf{HySys}\|_{\mathfrak{H}}\mathsf{CoDec}$ *satisfies that for all* $t \in \mathbb{R}_{\geq 0}$*,* $\sup_{(\hat{x},\bar{u})\in K(\phi_d(t))} \mathfrak{C}(\phi(t), \hat{x}) \leq \epsilon$*, where* $\phi_f \doteq (\phi, \phi_c, \phi_d) : \mathbb{R}_{\geq 0} \to X \times X_c \times X_d$ *is any trajectory of* $\mathsf{HySys}\|_{\mathfrak{H}}\mathsf{CoDec}$ *and* $\bar{u}$ *is the unique element in* U*.*

Definition 1.72 (Stabilization by a coder–decoder)**.** *Consider a hybrid system* $\mathsf{HySys} = (X, X_0, U, F, G)$ *and a cost function* $\mathfrak{C} : X \to \mathbb{R}_{\geq 0}$*. Let* $\mathsf{CoDec} =$

$(\mathsf{HySys_c}, \Upsilon, \mathsf{HySys_d}, K)$, $\mathsf{HySys_c} = (X_\mathrm{c}, X_\mathrm{c0}, X, F_\mathrm{c}, G_\mathrm{c})$, $\mathsf{HySys_d} = (X_\mathrm{d}, X_\mathrm{d0}, Y_\mathrm{t}\times Y, F_\mathrm{d}, G_\mathrm{d})$*, be a coder–decoder for* HySys *with universal output map* $\mathfrak{H} : X \rightrightarrows Y$*. We say that* CoDec stabilizes HySys *with respect to* $\mathfrak{C}$ *(via* $\mathfrak{H}$*) if the feedback composition* $\mathsf{HySys}\|_{\mathfrak{H}}\mathsf{CoDec}$ *is stable with respect to the cost function* $\mathfrak{C}_\mathrm{f} : X \times X_\mathrm{c} \times X_\mathrm{d} \to \mathbb{R}_{\geq 0}$ *defined by* $\mathfrak{C}_\mathrm{f}(x, x_\mathrm{c}, x_\mathrm{d}) = \mathfrak{C}(x)$*.*

Finally, we define the data rate of a coder–decoder, which accounts for the number of bits per unit of time that is needed to describe to output of the transmission map.

Definition 1.73 (Data rate of a coder–decoder)**.** *Let* CoDec *be a coder–decoder with period* T_t *and symbol set* Y_t*. The* data rate *of* CoDec*, denoted by* $\mathcal{R}(\mathsf{CoDec})$*, is defined as*

$$\mathcal{R}(\mathsf{CoDec}) = \frac{\lceil \log_2 |Y_\mathrm{t}| \rceil}{T_\mathrm{t}}$$

where $\lceil \cdot \rceil$ *is the* ceil *function and* $|Y_\mathrm{t}|$ *is the cardinality of* Y_t*.*

The data rate of a coder–decoder accounts for the channel capacity required for the communication between the coder and the decoder. Different coders–decoders, with different data rates, can be used for the state estimation or stabilization of a same hybrid system. Thus, given a hybrid system, one may wonder what is the minimal channel capacity required for state estimation or stabilization. This leads to the notions of minimal data rate for state estimation and stabilization of a hybrid system.

Definition 1.74 (Minimal data rate for state estimation)**.** *Consider an autonomous hybrid system* $\mathsf{HySys} = (X, X_0, U, F, G)$*, a cost function* $\mathfrak{C} : X \times X \to \mathbb{R}_{\geq 0}$ *and a universal output map* $\mathfrak{H} : X \rightrightarrows Y$*. The* minimal[17] data rate for state estimation *of* HySys *with respect to* $\mathfrak{C}$ *(and via* $\mathfrak{H}$*), denoted by* $\mathcal{R}_\mathrm{est}(\mathsf{HySys}, \mathfrak{C}, \mathfrak{H})$*, is defined as*

$$\mathcal{R}_\mathrm{est}(\mathsf{HySys}, \mathfrak{C}, \mathfrak{H}) = \sup_{\epsilon>0} \inf_{\mathsf{CoDec}} \mathcal{R}(\mathsf{CoDec}),$$

where the infimum is over all coders–decoders CoDec *that* ϵ*-observe* HySys *with respect to* $\mathfrak{C}$ *(via* $\mathfrak{H}$*).*

Definition 1.75 (Minimal data rate for stabilization)**.** *Consider a hybrid system* $\mathsf{HySys} = (X, X_0, U, F, G)$*, a cost function* $\mathfrak{C} : X \to \mathbb{R}_{\geq 0}$ *and a universal output map* $\mathfrak{H} : X \rightrightarrows Y$*. The* minimal[18] data rate for stabilization *of* HySys *with respect to* $\mathfrak{C}$ *(and via* $\mathfrak{H}$*), denoted by* $\mathcal{R}_\mathrm{stab}(\mathsf{HySys}, \mathfrak{C}, \mathfrak{H})$*, is defined as*

$$\mathcal{R}_\mathrm{stab}(\mathsf{HySys}, \mathfrak{C}, \mathfrak{H}) = \inf_{\mathsf{CoDec}} \mathcal{R}(\mathsf{CoDec}),$$

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[17] A more accurate name would be "*infimal* data rate", but the terminology "minimal data rate" has become standard in the literature.

[18] Same comment as in Definition 1.74.

where the infimum is over all coders–decoders CoDec *that stabilize* HySys *with respect to* $\mathfrak{C}$ *(via* $\mathfrak{H}$*).*

Note that the data rate of the coder–decoder only gives a lower bound on the channel capacity. Other aspects must also be taken into account in the implementation of the communication protocol. For instance: How are the symbols encoded and how are they sent? How do we account for possible delays and dropouts during the transmission? In this thesis, we mainly focus on the question of the minimal amount of information that one needs to observe or control hybrid systems, so that the notion of date rate plays an instrumental role in our analysis (see Chapter 3 on the quantized control of hybrid systems). However, we are also interested in the practical implementation of coders–decoders achieving optimal data rate bounds, and thus the description of such optimal coders–decoders will involve the description of the communication protocol and the assumptions on the communication channel.

1.5.2 Topological entropy

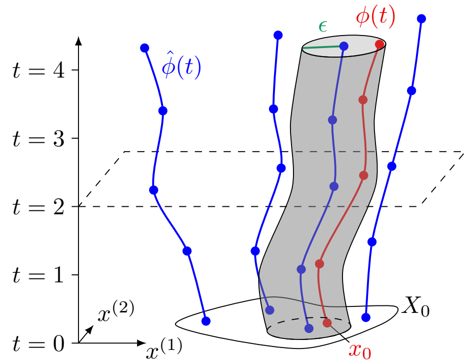
The notion of topological entropy, introduced by Adler et al. (1965), Dinaburg (1970) and Bowen (1971), accounts for the growth rate of the smallest number of functions necessary to approximate the trajectories of the system with arbitrary finite accuracy on bounded time intervals (with respect to the length of the interval). It can also be seen as the growth rate of the number of trajectories that are distinguishable with arbitrary finite accuracy on bounded time intervals. To formalize this, we first introduce the notions of spanning set and separated set. See also Figure 1.18 for an illustration.

Definition 1.76 (Spanning set)**.** *Consider an autonomous hybrid system* HySys $=(X,X_0,U,F,G)$ *and a cost function* $\mathfrak{C}:X\times X\to\mathbb{R}_{\geq 0}$*. Let* $\epsilon>0$ *and* $T\in\mathbb{R}_{\geq 0}$*. A set* $\mathcal{E}$ *of functions from* $[0,T)$ *to* X *(i.e.,* $\mathcal{E}\subseteq X^{[0,T)}$*) is said to be an* (ϵ,T)-spanning set *for* HySys *with respect to* $\mathfrak{C}$ *if for every trajectory* ϕ *of* HySys*, there is* $\hat{\phi}\in\mathcal{E}$ *such that for all* $t\in[0,T)$*,* $\mathfrak{C}(\phi(t),\hat{\phi}(t))\leq\epsilon$*. The smallest cardinality of an* (ϵ,T)*-spanning set for* HySys *with respect to* $\mathfrak{C}$ *is denoted by* $s_{\mathrm{span}}(\epsilon,T;\mathsf{HySys},\mathfrak{C})$ *(or* $s_{\mathrm{span}}(\epsilon,T)$ *if* HySys *and* $\mathfrak{C}$ *are clear from the context).*

Definition 1.77 (Separated set)**.** *Consider an autonomous hybrid system* HySys $=(X,X_0,U,F,G)$ *and a cost function* $\mathfrak{C}:X\times X\to\mathbb{R}_{\geq 0}$*. Let* $\epsilon>0$ *and* $T\in\mathbb{R}_{\geq 0}$*. A set* $\mathcal{F}$ *of functions from* $[0,T)$ *to* X *(i.e.,* $\mathcal{F}\subseteq X^{[0,T)}$*) is said to be an* (ϵ,T)-separated set *for* HySys *with respect to* $\mathfrak{C}$ *if each function in* $\mathcal{F}$ *is a (non-complete) trajectory of* HySys *and for any distinct* $\phi_1,\phi_2\in\mathcal{F}$*, there is* $t\in[0,T)$ *such that* $\mathfrak{C}(\phi_1(t),\phi_2(t))>\epsilon$*. The largest cardinality of an* (ϵ,T)*-separated set* *for* HySys *with respect to* $\mathfrak{C}$ *is denoted by* $s_{\text{sep}}(\epsilon, T; \mathsf{HySys}, \mathfrak{C})$ *(or* $s_{\text{sep}}(\epsilon, T)$ *if* HySys *and* $\mathfrak{C}$ *are clear from the context).*

In other words, two trajectories ϕ_1 and ϕ_2 are (ϵ, T)-separated if $\sup_{t\in[0,T)} \|\phi_1(t) - \phi_2(t)\| > \epsilon$.

Figure 1.18: $(\epsilon, 4)$-spanning set $\mathcal{E}$ (in blue) for a hybrid system HySys. Each trajectory ϕ (e.g., in red) of HySys is in the ϵ-tube around some function $\hat{\phi}$ of $\mathcal{E}$ for all time $t \in [0, 4)$.



Now, we introduce the concept of topological entropy.

Definition 1.78 (Topological entropy)**.** *Consider an autonomous hybrid system* HySys $= (X, X_0, U, F, G)$ *and a cost function* $\mathfrak{C} : X \times X \to \mathbb{R}_{\geq 0}$*. The* topological entropy *of* HySys *with respect to* $\mathfrak{C}$*, denoted by* $h_{\text{top}}(\mathsf{HySys}, \mathfrak{C})$ *(or* $h_{\text{top}}(\mathsf{HySys})$ *if* $\mathfrak{C}$ *is clear from the context), is defined as*

$$h_{\text{top}}(\mathsf{HySys}, \mathfrak{C}) = \sup_{\epsilon>0} \limsup_{T\to\infty} \frac{1}{T} \log_2 s_{\text{span}}(\epsilon, T; \mathsf{HySys}, \mathfrak{C}).$$

Example 1.6. Consider the continuous-time dynamical system Sys $= (\mathbb{R}, f)$, with $f(x) = x$ and with initial set $X_0 = [0, 1]$, and consider the cost function $\mathfrak{C}(x) = |x|$. For any $\epsilon > 0$ and $T \in \mathbb{R}_{\geq 0}$, one can show (see, e.g., Example 3.1 in Subsection 3.2.1 for a similar problem) that

$$(\epsilon^{-1}e^T - 1)/2 \leq s_{\text{span}}(\epsilon, T) \leq \epsilon^{-1}e^T + 1.$$

Hence, for $\epsilon > 0$ fixed, it holds that $\limsup_{T\to\infty} \frac{1}{T} \log_2 s_{\text{span}}(\epsilon, T) = \log_2(e)$. Since the latter is independent from ϵ, we find that $h_{\text{top}}(\mathsf{Sys}) = \log_2(e)$. The property that the limit superior does not depend on ϵ is typical for linear systems; for nonlinear systems, the limit superior may be increasing when $\epsilon \to 0$ but in general it is nevertheless bounded so that the topological entropy is still finite.

The following properties of the topological entropy are elementary but useful in the analysis of the topological entropy and its link with the minimal data rate for state observation.

Proposition 1.79 (Topological entropy and separated sets)**.** *Consider an autonomous hybrid system* $\mathsf{HySys} = (X, X_0, U, F, G)$ *and a cost function* $\mathfrak{C} : X \times X \to \mathbb{R}_{\geq 0}$*. It holds that*

$$h_{\text{top}}(\mathsf{HySys}, \mathfrak{C}) \leq \sup_{\epsilon>0} \limsup_{T\to\infty} \frac{1}{T} \log_2 s_{\text{sep}}(\epsilon, T; \mathsf{HySys}, \mathfrak{C}).$$

Moreover, if $\mathfrak{C}$ *satisfies the weak triangle inequality*[19]*, then the above inequality is an equality.*

Proof. See, e.g., Matveev and Savkin (2009, Lemma 2.3.8) for a proof of the first item with the cost $(x_1, x_2) \mapsto \|x_1 - x_2\|_\infty$. The case of a general cost function is identical. □

The topological entropy is a lower bound on the minimal data rate for state estimation of hybrid systems. Furthermore, if the initial set is forward invariant, then the topological entropy is also equal to the minimal data rate for state estimation. The reason for this is that given an accuracy $\epsilon > 0$ and a time horizon $T \in \mathbb{T}_{>0}$, one can use a "catalogue" of $s_{\text{span}}(\epsilon; T)$ functions that approximate the state of the system with accuracy ϵ on the time interval $[0, T)$ and thus send about $\log_2 s_{\text{span}}(\epsilon; T)$ bits of information to estimate the state of the system with accuracy ϵ until time T. Now, since the initial set is forward invariant, the process can be repeated for all subsequent time epochs of duration T. In the end, one will thus need to send about $\log_2 s_{\text{span}}(\epsilon; T)$ bits of information every T units of time, thereby ending up with a data rate of about $\frac{1}{T} \log_2 s_{\text{span}}(\epsilon; T)$; when $T \to \infty$, the data rate converges to the topological entropy.

Proposition 1.80 (Topological entropy and minimal data rate for state estimation)**.** *Consider an autonomous hybrid system* $\mathsf{HySys} = (X, X_0, U, F, G)$ *and a cost function* $\mathfrak{C} : X \times X \to \mathbb{R}_{\geq 0}$*. Assume that* $\mathfrak{H}$ *is the empty map* $\mathfrak{H} : X \rightrightarrows \emptyset$*. It holds that* $\mathcal{R}_{\text{est}}(\mathsf{HySys}, \mathfrak{C}, \mathfrak{H}) \geq h_{\text{top}}(\mathsf{HySys}, \mathfrak{C})$*. Moreover, if* $X_0 = X$*, then* $\mathcal{R}_{\text{est}}(\mathsf{HySys}, \mathfrak{C}, \mathfrak{H}) = h_{\text{top}}(\mathsf{HySys}, \mathfrak{C})$*.*

Proof. See, e.g., Matveev and Pogromsky (2016, Theorem 8) for a proof with the cost $(x_1, x_2) \mapsto \|x_1 - x_2\|$. The case of a general cost function is along the same lines, and left to the reader. □

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[19]This means that there is a class-$\mathcal{K}$ function α such that for any $x_1, x_2, x_3 \in X$, $\mathfrak{C}(x_1, x_3) \leq \alpha(\mathfrak{C}(x_1, x_2) + \mathfrak{C}(x_2, x_3))$.

Chapter 2

Dominance analysis of hybrid systems

In the first chapter, we introduced the definitions and concepts related to the analysis and control of hybrid systems. In this chapter, we present the first part of our contributions, which deals with the study of the concept of dominance for switched linear systems and smooth dynamical systems. We will see that this concept connects with several other concepts in systems and control theory, such as positivity, hyperbolicity (introduced in Subsection 1.2.3) and topological entropy (introduced in Subsection 1.5.2 and thoroughly studied for switched linear systems in the next chapter).

2.1 Introduction and literature review

The system property of having a separation of their (linearized) dynamics into a dominant component and a dominated component has proved instrumental in various areas of systems and control theory. To illustrate this and to introduce the topic, let us start with an overview of system classes satisfying this property and their applications; this will also be the opportunity to review the relevant literature.

Positive systems are linear systems that leave a solid convex pointed cone invariant; that is, linear systems for which there is a conic (and convex, with nonempty interior and containing no bi-infinite line) region of the state space such that the trajectories of the system starting in that region stay in it for all positive times. A classical example of such systems are linear systems whose matrix has only positive entries (in the discrete-time case) or only positive off-diagonal entries (in the continuous-time case). Indeed, for these systems, it holds that the trajectories starting in the nonnegative orthant stay in it for ever; see Figure 2.1 for an illustration. Positive systems appear naturally in a wide range of applications, such as economics, biology, Markov chains, opinion dynamics, etc. Therefore, they have been an important topic of research for some time now; see, e.g., Luenberger (1979), Berman et al. (1989), Kaczorek (2002) and Farina and Rinaldi (2000) for surveys. It was soon realized that the property of cone invariance significantly restricts the behavior of the system: namely, these systems have a single dominant eigenvector (called "Perron–Frobenius eigenvector") which is a 1-dimensional attractor for the system (see, e.g., Vandergraft, 1968). Consequently, positive systems allow for a simplified analysis and control of their dynamics; see, e.g., Luenberger (1979), Farina and Rinaldi (2000), Rantzer (2015) and references therein.

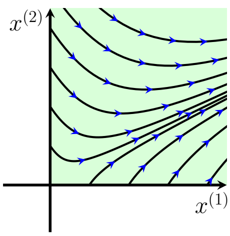


Figure 2.1: Positive continuous-time linear system. The trajectories starting in the positive orthant (in green) stay in it for all $t \geq 0$.

The concept of positive system has been generalized in several directions: namely, positive *time-varying* systems, i.e., linear time-varying systems leaving a solid convex pointed cone invariant (see, e.g., Parlett, 1970, and Pituk and Pötzsche, 2019); *monotone* systems, i.e., smooth dynamical systems whose prolonged dynamics leaves a solid convex pointed cone invariant (see, e.g., Smith, 1995, Angeli and Sontag, 2003, and Hirsch and Smith, 2006); and more recently, *path-complete positive* switched linear systems (Forni et al., 2017) and *differentially positive* systems (Forni and Sepulchre, 2016) which further extend the property of cone invariance by moving from a single cone to a family of convex pointed cones. These generalizations enjoy similar properties as positive systems: in particular, their asymptotic behavior lies in a 1-dimensional object. This property is also known as *weak ergodicity* in the case of linear systems (e.g., positive time-varying systems or path-complete positive switched linear systems), and translates by the fact that the normalized trajectories of the system are *incrementally stable*, meaning that the normalized trajectories converge to each other, independently of their initial condition; see Figure 2.2 for an illustration. This fundamental property has been used in a large number of contexts, e.g., for the analysis of Markov chains (see, e.g., Seneta, 1981), population dynamics (see, e.g., Parlett, 1970, and Golubitsky et al., 1975), or communication networks (see, e.g., Shorten et al., 2006).

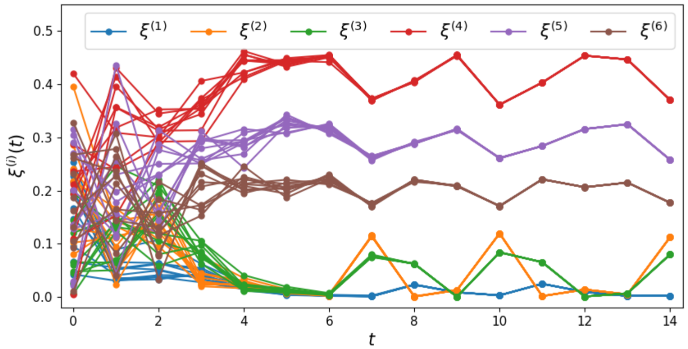


Figure 2.2: Weak ergodicity. The normalized trajectories of positive time-varying linear systems or path-complete positive switched linear systems, starting from different initial conditions, converge to each other as $t \to \infty$. This can also be seen as the incremental stability property of the system in the projective space.

Recently, the concept of *p-dominance* was introduced by Forni and Sepulchre (2019) to generalize the approach of positive or monotone systems to study dynamical systems whose linearized dynamics can be split into a p-dimensional dominant component and a complementary $(n-p)$-dimensional *stable* component (with n the dimension of the system). In particular, they study continuous-time dynamical systems whose linearized dynamics leaves a *quadratic p-cone* invariant. In this sense, the property of p-dominance extends the classical notion of positivity by introducing cones that are compatible with p-dimensional asymptotic behavior. For instance, it is shown that this property implies that the system has an asymptotic behavior that is essentially the one of a p-dimensional system; this allows to study complex phenomenons like multi-stability or limit cycles for high-dimensional dynamical systems. The use of quadratic p-cones allows to encode the invariance property of the cone as the feasibility of a set of matrix inequalities, leading to efficient methods for the computation of an invariant quadratic p-cone for p-dominance. Note that this property has also been used in the context of positivity, to compute invariant convex pointed quadratic cones; see, e.g., Hildebrand (2007) and Grussler and Rantzer (2014).

The property of having a separation of the dynamics also appears in the notions of *partial hyperbolicity* and *exponential dichotomy* (see, e.g., Brin and Pesin, 1974, and Barreira and Valls, 2008), describing dynamical systems whose linearized dynamics can be split into two components: a p-dimensional dominant component growing exponentially faster than a complementary $(n-p)$-dimensional dominated component (with n the dimension of the system). The separation of the dynamics is described using the concept of *dominated splitting* (introduced by Mañé, 1987, in its celebrated work on the stability conjecture). Partially hyperbolic systems have received a lot of attention in the literature, namely for their applications in the study of global properties of dynamical systems, such as chaos, topological entropy, structural stability, etc. (see, e.g., Brin and Pesin, 1974, Hirsch et al., 1977, Barreira and Valls, 2008, and Pesin, 2004; see also Subsection 1.2.3 for the definition and properties of hyperbolic systems).

In this thesis, we extend the concepts of p-dominance and dominated splittings to *discrete-time switched linear systems* and *discrete-time smooth dynamical systems*. This allows us to provide mathematical and algorithmic frameworks to study the property of having a separation of the (linearized) dynamics for these paradigmatic classes of (hybrid) systems.

Dominance and dominated splittings for discrete-time switched linear systems

First, we focus on discrete-time switched linear systems. Thriving on ideas from path-complete Lyapunov theory (see Subsection 1.3.3), we extend the property of p-dominance by moving from a single quadratic p-cone to a family of quadratic p-cones whose invariance properties are driven by an automaton capturing the admissible switches of the system. The goal is to increase the expressiveness of the property of p-dominance while preserving the feature of a separation of the dynamics. In particular, we show that this notion of p-dominance provides a necessary and sufficient condition for the system to have a dominated p-splitting, that is, for its dynamics to be decomposable into a p-dimensional dominant dynamics growing exponentially faster than a complementary $(n-p)$-dimensional dominated dynamics (with n the dimension of the system).[1]

We also provide an algorithmic framework for the verification of the property of p-dominance for discrete-time switched linear systems. For that, we leverage the algorithmic approach used in Forni and Sepulchre (2019) (relying on matrix inequalities to encode the invariance properties of the quadratic p-cones) and, with the help of advanced results from linear algebra and automata

[1] Let us mention that the notions of p-dominance and dominated splitting considered here are slightly more general than the ones in Forni and Sepulchre (2019), in that we do not require the dominated dynamics to be stable. The goal is to capture a larger class of systems (though at the cost of deriving weaker general properties of these p-dominant systems), but we also discuss and characterize the case where the dominated dynamics is stable (see also Remark 2.1 in Subsection 2.2.2). For the sake of avoiding unnecessarily heavy terminology, we keep the term *p-dominance* to refer to this more general concept of dominance.

theory, we extend it for the computation of families of quadratic p-cones whose invariance properties are driven by an automaton.

Dominance and dominated splittings for discrete-time smooth dynamical systems

Secondly, we introduce the notion of p-dominance for discrete-time smooth dynamical systems. As for p-dominant continuous-time smooth dynamical systems (Forni and Sepulchre, 2019), the approach is differential: the criterion of p-dominance is formulated on the prolonged system and accounts for the fact that the associated linear system, which can be seen as a switched linear system with an infinite set of modes, is p-dominant. Thriving on results from dominance analysis of switched linear systems, we show that p-dominant discrete-time smooth dynamical systems admit a dominated p-splitting, meaning that their linearized dynamics can be decomposed into a p-dimensional dominant dynamics growing exponentially faster than a complementary $(n-p)$-dimensional dominated dynamics (with n the dimension of the system); this allows to study behaviors like partial hyperbolicity or exponential dichotomy for these systems.

Furthermore, thriving on the algorithmic framework for the verification of p-dominance for switched linear systems and on the technique of abstraction for dynamical systems (see Subsection 1.4.1)—allowing to abstract the prolonged system as a switched linear system with an infinite set of modes—, we provide an algorithmic framework for the verification of the property of p-dominance for discrete-time smooth dynamical systems.

Applications: incremental stability, hyperbolicity, quantized control, etc.

Finally, we present several applications, supported by numerical examples, of the theory of p-dominance for discrete-time switched linear systems and discrete-time smooth dynamical systems. Namely, we show that the theory of p-dominant switched linear systems can be used to analyze the convergence of the trajectories of such systems to a low-dimensional time-varying attractor (incremental stability in the projective space; see above), and present applications in population dynamics. It can also be used to obtain bounds on the topological entropy of these systems (which will be useful in the next chapter, on the quantized control of switched linear systems). As for discrete-time smooth dynamical systems, we show that the theory of p-dominance can be used for the formal verification of the property of hyperbolicity (introduced in Subsection 1.2.3) and to obtain bounds on the topological entropy of these systems (used in quantized control as well).

The results presented in this chapter have been reported in

- Guillaume O Berger, Fulvio Forni, and Raphaël M Jungers. Path-complete p-dominant switching linear systems. In *2018 IEEE 57th IEEE Conference on Decision and Control (CDC)*, pages 6446–6451. IEEE, 2018. doi: 10.1109/CDC.2018.8619703.

- Guillaume O Berger and Raphaël M Jungers. A converse Lyapunov theorem for p-dominant switched linear systems. In *2019 18th European Control Conference (ECC)*, pages 1263–1268. IEEE, 2019. doi: 10.23919/ECC.2019.8795923.

- Guillaume O Berger and Raphaël M Jungers. Formal methods for computing hyperbolic invariant sets for nonlinear systems. *IEEE Control Systems Letters*, 4(1):235–240, 2020c. doi: 10.1109/LCSYS.2019.2923923.

- Guillaume O Berger and Raphaël M Jungers. p-dominant switched linear systems. *Automatica*, 132:109801, 2021b. doi: 10.1016/j.automatica.2021.109801.

The algorithms presented in this chapter are implemented in the Julia package: `https://github.com/guberger/Dominance.jl`. This package was used among others to produce the figures and to analyze the numerical examples presented in this chapter.

2.2 Dominance analysis of switched linear systems

In this section, we introduce the concept of p-dominance for discrete-time switched linear systems. As explained in Section 2.1, we combine ideas from p-dominance of continuous-time smooth dynamical systems and from path-complete Lyapunov theory to formulate a criterion of p-dominance based on a family of quadratic p-cones whose contraction properties are driven by an automaton accepting every switching signal of the system. We show that discrete-time switched linear systems are p-dominant if and only if they admit a dominated p-splitting, that is, if and only if their dynamics can be decomposed into a p-dimensional dominant component and an $(n-p)$-dimensional dominated component (with n the dimension of the system) for every switching signal. We also provide an algorithmic framework for the verification of the property of

p-dominance of these systems. Finally, we discuss some applications of the theory of p-dominant discrete-time switched linear systems, namely for the study of population dynamics and for the estimation of the topological entropy of these systems.

The section is organized as follows. In Subsection 2.2.2, we introduce the main concepts related to p-dominance of discrete-time switched linear systems and the characterization of their asymptotic behavior. In Subsection 2.2.3, we describe the algorithmic framework for the verification of p-dominance of discrete-time switched linear systems. Finally, numerical examples and examples of application are presented in Subsection 2.2.4.

Notation. In this section, all considered switched linear systems are discrete-time switched linear systems, and thus for the sake of brievety, we will refer to them simply as *switched linear systems.* For $A \subseteq \mathbb{R}^n$, $\operatorname{int} A$ denotes the interior of A. If $A \subseteq \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$, then MA is the image of A by M, i.e., $MA = \{Mx : x \in A\}$. The set of symmetric matrices in $\mathbb{R}^{n\times n}$ is denoted by $\mathbb{S}^{n\times n}$. For $P, Q \in \mathbb{S}^{n\times n}$, we write $P \succ Q$ (resp. $P \succeq Q$) if $P - Q$ is positive definite (semidefinite).

2.2.1 Warm-up: p-dominant LTI systems

We start our analysis with the classical case[2] of linear time-invariant (LTI) systems. Therefore, let us consider a LTI system[3] $\mathsf{Sys} = (\mathbb{R}^n, A)$, where $A \in \mathbb{R}^{n\times n}$. For a reminder, the trajectories $\xi : \mathbb{N} \to \mathbb{R}^n$ of Sys satisfy $\xi(t+1) = A\xi(t)$ for all $t \in \mathbb{N}$. In this section and in the rest of this chapter, p is a fixed integer between 0 and n, called the *degree of dominance.*

First, let us remind the definition of inertia of a symmetric matrix.

Definition 2.1 (Inertia of a symmetric matrix). *Let $P \in \mathbb{S}^{n\times n}$ and $k \in \{0, \dots, n\}$. The matrix P is said to have* inertia $(k, 0, n-k)$ *if it has k negative eigenvalues and $n-k$ positive eigenvalues. The set of matrices of $\mathbb{S}^{n\times n}$ with inertia $(k, 0, n-k)$ is denoted by $\mathbb{S}^{n\times n}_k$.*

This allows us to define the notion of p-dominant LTI system.

Definition 2.2 (p-dominant LTI system). *A LTI system* $\mathsf{Sys} = (\mathbb{R}^n, A)$ *is said to be p-*dominant *(or* dominant with degree p*) if there is $\gamma > 0$ and a matrix*

[2]The results presented for the LTI case are merely reformulations of commonly known results, in order to introduce the fundamental concepts that will be used in the study of p-dominance for switched linear systems.

[3]Thus, in our notation, LTI systems are understood as dynamical systems (see Definition 1.6 in Subsection 1.1.2). Alternatively, they could be described as switched linear systems with one mode, but this would make the notation a bit longer, namely $\mathsf{Sys} = (\mathbb{R}^n, \{A\})$.

$P \in \mathbb{S}_p^{n\times n}$ *such that*

$$A^\intercal P A - \gamma^2 P \prec 0. \tag{2.1}$$

The parameter γ in Definition 2.2 is called a *separation rate* (or simply *rate*) of p-dominance of Sys. The reason for this name is made clear in the following proposition:

Proposition 2.3. *A LTI system* $\mathsf{Sys} = (\mathbb{R}^n, A)$ *is p-dominant with rate* $\gamma > 0$ *if and only it satisfies any (and thus all) of the following conditions:*

1. *The matrix A has p eigenvalues with modulus $|\lambda_i| > \gamma$, and $n-p$ eigenvalues with modulus $|\lambda_i| < \gamma$;*

2. *There is a splitting of the state space $\mathbb{R}^n = E^{\mathrm{s}} \oplus E^{\mathrm{u}}$, where E^{s} is a linear subspace of dimension $n-p$ and E^{u} is a linear subspace of dimension p satisfying that (i) $AE^{\mathrm{s}} \subseteq E^{\mathrm{s}}$ and $AE^{\mathrm{u}} = E^{\mathrm{u}}$, and (ii) there is $C \geq 1$ and $\mu \in (0,1)$ such that*

   - *for every $x \in E^{\mathrm{s}}$ and $t \in \mathbb{N}$, $\|A^t x\| \leq \sqrt{C\mu^t}\,\gamma^t\|x\|$;*
   - *for every $x \in E^{\mathrm{u}}$ and $t \in \mathbb{N}$, $\|A^t x\| \geq \dfrac{1}{\sqrt{C\mu^t}}\,\gamma^t\|x\|$.*

Proof. See Appendix A.2.2. □

Condition 2 in Proposition 2.3 implies that for any $x_1 \in E^{\mathrm{s}}$ and $x_2 \in E^{\mathrm{u}} \setminus \{0\}$,

$$\frac{\|A^t x_1\|}{\|A^t x_2\|} \leq \frac{\|x_1\|}{\|x_2\|} C\mu^t \qquad \text{for all } t \in \mathbb{N}. \tag{2.2}$$

Therefore, the pair $(E^{\mathrm{s}}, E^{\mathrm{u}})$ in Condition 2 in Proposition 2.3 is called a *dominated splitting* as it ensures a decomposition of the dynamics of the system into a p-dimensional *dominant* component and a complementary $(n-p)$-dimensional *dominated* component. In the case of LTI systems, E^{s} and E^{u} are the eigenspaces associated to the $n-p$ eigenvalues with modulus $< \gamma$, and the p eigenvalues with modulus $> \gamma$ respectively. Moreover, by Proposition 2.3, the rate γ gives a maximal and minimal rate of growth of the trajectories starting in E^{s} and E^{u}, respectively.

Remark 2.1. Let us mention that the notion of p-dominance considered in Forni and Sepulchre (2019) assumes that the dominated dynamics is stable. In this case, the asymptotic behavior of the system is dictated by the dominant dynamics only; this allows to derive even stronger results on the properties of the system. In this thesis, we consider a more general notion of p-dominance, requiring only that the dominant dynamics grows exponentially faster than the dominated dynamics; the goal being to capture a larger class of systems. But we also discuss and characterize the case where the dominated dynamics is stable (see Theorem 2.15 in Subsection 2.2.2; in the case of LTI systems, this is equivalent to requiring that $\gamma < 1$ in Definition 2.2), which is instrumental for instance for the characterization of hyperbolicity and for applications in quantized control.

Letting $V : \mathbb{R}^n \to \mathbb{R}$ be defined by $V(x) = x^\top P x$, the dissipation inequality (2.1) can be read as follows: for every trajectory $\xi : \mathbb{N} \to \mathbb{R}^n$ of Sys, it holds that $V(\xi(t+1)) \leq \gamma^2 V(\xi(t)) - \epsilon\|\xi(t)\|^2$ for all $t \in \mathbb{N}$ and for some $\epsilon > 0$. This implies that the *quadratic p-cone* defined as the nonpositive sublevel set of V is *contracted* by the system. This property is formalized in Proposition 2.5 below; first, let us introduce the notion of quadratic cone associated to a symmetric matrix.

Definition 2.4 (Quadratic p-cone)**.** *Let $P \in \mathbb{S}_p^{n\times n}$. The* quadratic p-cone *(or* cone*) associated to P, denoted by $\mathcal{K}(P)$, is defined as $\mathcal{K}(P) = \{x \in \mathbb{R}^n : x^\top P x \leq 0\}$.*

See Figure 2.3 for an illustration. It is easily seen that $\mathcal{K}(P)$ is a cone, meaning that for any $\alpha \in \mathbb{R}_{\geq 0}$, $\alpha\mathcal{K}(P) \subseteq \mathcal{K}(P)$. The prefix "$p$" accounts for the fact that the largest dimension of a linear subspace contained in $\mathcal{K}(P)$ is p. The above allows to formulate the following geometric characterization of p-dominant LTI systems, equivalent to (2.1).

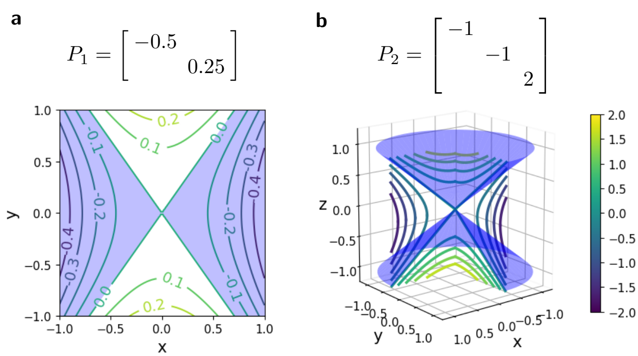


Figure 2.3: **a**: Level sets of $V : \mathbb{R}^2 \to \mathbb{R} : x \mapsto x^\top P_1 x$. $\mathcal{K}(P_1)$ is the quadratic 1-cone ($P_1 \in \mathbb{S}_1^{2\times 2}$) represented by the region in blue. **b**: Level sets of $V : \mathbb{R}^3 \to \mathbb{R} : x \mapsto x^\top P_2 x$. $\mathcal{K}(P_2)$ is the quadratic 2-cone ($P_2 \in \mathbb{S}_2^{3\times 3}$) consisting in the nonconvex region that contains the negative level sets of V, i.e., $\mathcal{K}(P_2)$ is the whole space except the two "ice-cream cones" delimited by the surface in blue.

Proposition 2.5. *A LTI system* Sys $= (\mathbb{R}^n, A)$ *is p-dominant if and only if there is* $P \in \mathbb{S}_p^{n\times n}$ *such that the associated quadratic p-cone* $\mathcal{K}(P)$ *is contracted into itself by* Sys*, that is,*

$$A(\mathcal{K}(P) \setminus \{0\}) \subseteq \operatorname{int} \mathcal{K}(P). \tag{2.3}$$

Proof. See Appendix A.2.3. □

The above concepts and results are illustrated in the example below. In particular, the eigenvalue plots, depicted in Figure 2.4-a, will be instrumental in the analysis and computation of p-dominance for switched linear systems (see Subsections 2.2.3 and 2.2.4).

Example 2.1. Consider the LTI system Sys $= (\mathbb{R}^2, A)$ with $A = \left[\begin{smallmatrix} \alpha & \alpha-1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\alpha = 0.2$. The eigenvalues of A satisfy $|\lambda_1| > \gamma > |\lambda_2|$ with $\gamma = 0.6$; see Figure 2.4-a. Thus, Sys is 1-dominant with rate γ (Proposition 2.3). It follows that (2.1) holds with some $P \in \mathbb{S}_1^{2\times 2}$. The quadratic 1-cone $\mathcal{K}(P)$ associated to such a P is represented in Figure 2.4-b. We observe that $\mathcal{K}(P)$ is contracted into itself by A, as predicted by Proposition 2.5.

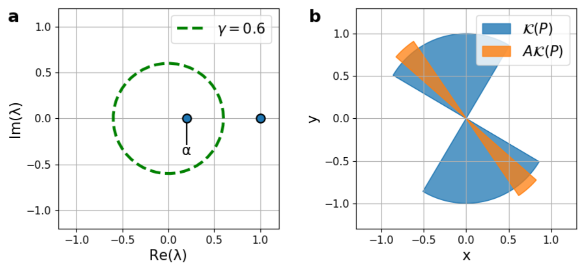


Figure 2.4: **a**: Eigenvalues of A (see Example 2.1). **b**: Quadratic 1-cone $\mathcal{K}(P)$ and its image by A.

Remark 2.2. For related results to the geometric characterization of p-dominance (Proposition 2.5), see, e.g., Stern and Wolkowicz (1991) where it is shown that a LTI system admits a pointed invariant ellipsoidal cone if and only if it has a positive eigenvalue strictly larger in modulus than any other eigenvalue; this is thus a particular case of 1-dominance where the dominant eigenvector is positive, translated by the fact that the associated cone is pointed (unlike quadratic 1-cones which consist in two "ice-cream cones"). Important classes of LTI systems satisfying the eigenvalue separation property of p-dominance include *relaxation systems* (see, e.g., Willems, 1976, and Pates et al., 2019), and *totally positive systems* (see, e.g., Margaliot and Sontag, 2019, Grussler and Sepulchre, 2020, and Grussler et al., 2021); indeed, for these systems, it holds that $\lambda_1 > \lambda_2 > \ldots > \lambda_n \geq 0$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A (see, e.g., Willems, 1976, Theorem 4, and Margaliot and Sontag, 2019, Theorem 1).

In the next subsection, we extend the notion of p-dominance to switched linear systems. Our goal is to characterize such systems that have a p-dimensional asymptotic behavior; a property formalized with a condition similar to the domination relation (2.2) (see Theorem 2.11 in the next subsection). Unlike LTI systems, this property cannot be deduced from an eigenvalue decomposition of the system. For this reason, the definition of p-dominant switched linear systems relies on dissipation inequalities involving symmetric matrices, like (2.1).

2.2.2 p-dominant switched linear systems

In this subsection, we extend the property of p-dominance to switched linear systems. Therefore, let us consider a switched linear system[4] $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$. For a reminder, the trajectories $(\xi, \sigma) : \mathbb{N} \to \mathbb{R}^n \times \Sigma$ of SwS satisfy $\xi(t+1) = A_{\sigma(t)}\xi(t)$ for all $t \in \mathbb{N}$, where $\xi : \mathbb{N} \to \mathbb{R}^n$ is the *continuous variable* and $\sigma : \mathbb{N} \to \Sigma$ is the *switching signal* of the trajectory, which specifies the *mode* $i \in \Sigma$ of SwS at each time $t \in \mathbb{N}$. In applications, it is sometimes useful to restrict the set of admissible switching signals (see, e.g., Section 2.1). The set of admissible complete switching signals of SwS is denoted by $\mathcal{S}(\mathsf{SwS})$, or $\mathcal{S}$ if SwS is clear from the context (see Definition 1.34 in Subsection 1.3.1).

Definition of path-complete p-dominance

In Subsection 1.3.3, we introduced the concept of path-complete Lyapunov functions for the stability analysis of continuous-time and discrete-time switched systems (see Definition 1.53). The definition of a path-complete Lyapunov function relies on a timed automaton accepting every switching signal of the system and on a set of "energy" functions whose decrease properties with respect to the system are dictated by the timed automaton. We will use a similar approach for the definition of the property of path-complete p-dominance for switched linear systems.

Since we restrict our attention to discrete-time systems, we will use timed automata whose time set is equal $\{1\}$ and thus refer to them simply as "automata". For the sake of completeness, we remind below the relevant notions related to timed automata, in this specific case.

Definition 2.6 (Automaton)**.** *An* automaton *is a triplet* (Q, Σ, Θ) *where* Q *is a finite set, called the set of* states*,* Σ *is a finite set, called the* alphabet*, and* $\Theta \subseteq Q \times \Sigma \times Q$ *is a finite set of* transitions*.*

---

[4]We refer the reader to Section 1.3 for the notation and definitions related to switched systems.

For a transition $\theta = (q_1, i, q_2) \in \Theta$, we denote its *source* q_1 by $\mathsf{s}(\theta)$, its *target* q_2 by $\mathsf{t}(\theta)$, and its *mode* i by $\mathsf{i}(\theta)$. A *path* of length T, where $T \in \mathbb{N} \cup \{\infty\}$, in an automaton (Q, Σ, Θ) is any sequence $(\theta_t)_{t=0}^{T-1} \subseteq \Theta$ satisfying that $\mathsf{t}(\theta_t) = \mathsf{s}(\theta_{t+1})$ for all $t \in \{0, \dots, T-2\}$.

Definition 2.7 (Admissible signal for an automaton)**.** *Consider an automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$*. A function* $\sigma : \mathbb{N} \to \Sigma$ *is said to be* admissible *for* Aut *(or* accepted *by* Aut*) if there is a path* $(\theta_t)_{t=0}^{\infty}$ *in* Aut *satisfying that for all* $t \in \mathbb{N}$, $\sigma(t) = \mathsf{i}(\theta_t)$.

The notions of automaton and accepted signals are illustrated in Figure 2.5.

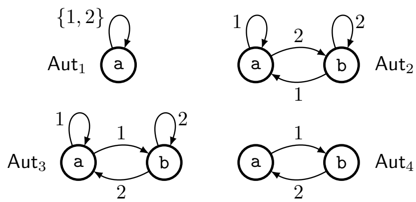


Figure 2.5: Four automata with $\Sigma = \{1, 2\}$, and $Q = \{\mathsf{a}\}$ (for Aut_1) or $Q = \{\mathsf{a}, \mathsf{b}\}$ (for Aut_2, Aut_3 and Aut_4). The transitions are represented by the edges (i.e., $q_1 \xrightarrow{i} q_2$ if and only if $(q_1, i, q_2) \in \Theta$). Aut_1, Aut_2 and Aut_3 accept every switching signal $\sigma : \mathbb{N} \to \Sigma$, while Aut_4 accepts only the switching signals $\sigma : \mathbb{N} \to \Sigma$ that are a strict alternation of "1" and "2".

The property of path-complete p-dominance, introduced below, extends the approach of cone invariance, used in the analysis of p-dominant LTI systems, by considering a *set* of quadratic p-cones whose contraction properties are driven by an automaton accepting every switching signal of the system. As for the LTI case, the quadratic p-cones are represented by symmetric matrices P_q with fixed inertia and the contraction properties are captured by matrix inequalities, similar to (2.1), depending on the transitions of the automaton and on a set of positive rates (one rate per transition).

Definition 2.8 (p-dominant switched linear system)**.** *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$*. We say that* SwS *is* path-complete p-dominant *(or* p-dominant*) if there is an automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$ *accepting every switching signal* $\sigma \in \mathcal{S}$*, a set of rates* $\{\gamma_\theta\}_{\theta\in\Theta} \subseteq \mathbb{R}_{>0}$ *and a set of matrices* $\{P_q\}_{q\in Q} \subseteq \mathbb{S}_p^{n\times n}$ *such that for every* $\theta \in \Theta$,

$$A_{\mathsf{i}(\theta)}^{\mathsf{T}} P_{\mathsf{t}(\theta)} A_{\mathsf{i}(\theta)} - \gamma_\theta^2 \, P_{\mathsf{s}(\theta)} \prec 0. \tag{2.4}$$

See Examples 2.2 and 2.3 below for illustrations.

Remark 2.3. Note that for a given automaton and a given set of rates, there is at most one value of p for which the system is p-dominant; see Proposition 2.19 in Subsection 2.2.3. However, depending on the automaton and the set of rates, the system can be p-dominant for different values of p.

As mentioned above, the dissipation inequalities (2.4) capture the fact that the quadratic p-cones $\{\mathcal{K}(P_q)\}_{q\in Q}$ are contracted by the system along the transitions of Aut; see Figure 2.6 for an illustration. This leads to the following equivalent characterization of p-dominant switched linear systems.

Proposition 2.9. *A switched linear system* $\mathsf{SwS} = (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *is p-dominant if and only if there is an automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$ *accepting every switching signal* $\sigma \in \mathcal{S}$ *and a set of matrices* $\{P_q\}_{q\in Q} \subseteq \mathbb{S}_p^{n\times n}$ *such that the associated quadratic p-cones* $\{\mathcal{K}(P_q)\}_{q\in Q}$ *are contracted into each other by* SwS *along the transitions of* Aut*, that is, for all* $\theta \in \Theta$*,*

$$A_{\mathsf{i}(\theta)}(\mathcal{K}(P_{\mathsf{s}(\theta)}) \setminus \{0\}) \subseteq \operatorname{int}\mathcal{K}(P_{\mathsf{t}(\theta)}). \tag{2.5}$$

Proof. Same as for Proposition 2.5. □

The examples below illustrate the concept of p-dominant switched linear systems and the contraction property of Proposition 2.9.

Example 2.2. Consider the switched linear system $\mathsf{SwS} = (\mathbb{R}^2, \{A_i\}_{i\in\Sigma})$ under arbitrary switching, with $\Sigma = \{1, 2\}$, $A_1 = \left[\begin{smallmatrix} 1 & 0 \\ 1-\alpha & \alpha \end{smallmatrix}\right]$, $A_2 = \left[\begin{smallmatrix} \alpha & \alpha-1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\alpha = 0.1$; which may occur for instance in the modeling of opinion dynamics with antagonistic interactions and switching topologies (see, e.g., Meng et al., 2016). This system is 1-dominant with the automaton Aut_2 presented in Figure 2.5 and with the set of rates $\{\gamma_\theta\}_{\theta\in\Theta}$ given by $\gamma_\theta = 0.32$ for all $\theta \in \Theta$. This means that there are $P_{\mathsf{a}}, P_{\mathsf{b}} \in \mathbb{S}_1^{2\times 2}$ satisfying (2.4) with this automaton and this set of rates. The quadratic 1-cones associated to P_{a} and P_{b} are represented in Figure 2.6. We observe that the cones satisfy the cone contraction property (2.5).

The selection of the value of the rates in Example 2.2 will be discussed in Example 2.5, after we have presented a set of constraints that must be satisfied by the set of rates (see Proposition 2.19 in Subsection 2.2.3). The verification of p-dominance of the system with this set of rates was achieved by using the algorithm described in Corollary 2.18; see Subsection 2.2.3 on the algorithmic aspects of p-dominance analysis for details.

Example 2.3. Consider the switched linear system $\mathsf{SwS} \sim (\mathbb{R}^2, \{A_i\}_{i\in\Sigma})$ with $\Sigma = \{1, 2\}$, $A_1 = \left[\begin{smallmatrix} 2 & \\ & 4 \end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix} 1 & \\ & \frac{1}{8} \end{smallmatrix}\right]$. Assume that the switching signals of SwS are constrained to be a strict alternation of "1" and "2". Then, the automaton Aut_4 in Figure 2.5 accepts every switching signal $\sigma \in \mathcal{S}$. We show

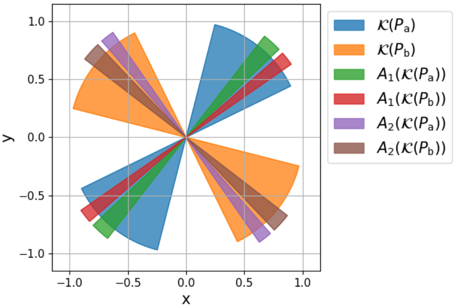


Figure 2.6: Quadratic 1-cones $\mathcal{K}(P_\mathtt{a})$ and $\mathcal{K}(P_\mathtt{b})$ and their images by A_1 and A_2 (see Example 2.2).

that the system is 1-dominant with this automaton and some set of rates. Indeed, consider the symmetric matrices $P_\mathtt{a} = \begin{bmatrix} -1 & \\ & 8 \end{bmatrix}$ and $P_\mathtt{b} = \begin{bmatrix} -\frac{1}{2} & \\ & \frac{1}{4} \end{bmatrix}$, which both belong to $\mathbb{S}_1^{2\times 2}$, and the set of rates $\{\gamma_\theta\}_{\theta\in\Theta}$ given by $\gamma_\theta = 1$ for all $\theta \in \Theta$. Then, the matrix inequality (2.4) is satisfied for every $\theta \in \Theta$. Indeed, for $\theta = (\mathtt{a}, 1, \mathtt{b})$, we get $A_1^\top P_\mathtt{b} A_1 - P_\mathtt{a} = \begin{bmatrix} -2 & \\ & 4 \end{bmatrix} - \begin{bmatrix} -1 & \\ & 8 \end{bmatrix} = \begin{bmatrix} -1 & \\ & -4 \end{bmatrix}$, and for $\theta = (\mathtt{b}, 2, \mathtt{a})$, we get $A_2^\top P_\mathtt{a} A_2 - P_\mathtt{b} = \begin{bmatrix} -1 & \\ & \frac{1}{8} \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} & \\ & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \\ & -\frac{1}{8} \end{bmatrix}$.

Asymptotic behavior of p-dominant switched linear systems

In this subsection, we show that p-dominant switched linear systems inherit the dynamical properties of p-dominant LTI systems, in the sense that their dynamics can be split into a dominant component and a dominated component. The difference with the LTI case is that for switched linear systems, the subspaces describing the decomposition of the dynamics are not fixed anymore, but may vary with time. To formalize this, we introduce the notion of time-varying splitting.

Definition 2.10 (Time-varying splitting of $\mathbb{R}^n$)**.** *A time-varying splitting (or* splitting*) of $\mathbb{R}^n$ is an ordered pair $(E^\mathrm{s}, E^\mathrm{u})$, where $E^\mathrm{s} : \mathbb{N} \rightrightarrows \mathbb{R}^n$ and $E^\mathrm{u} : \mathbb{N} \rightrightarrows \mathbb{R}^n$ are set-valued functions such that for all $t \in \mathbb{N}$, $E^\mathrm{s}(t)$ and $E^\mathrm{u}(t)$ are linear subspaces satisfying $\mathbb{R}^n = E^\mathrm{s}(t) \oplus E^\mathrm{u}(t)$. We say that $(E^\mathrm{s}, E^\mathrm{u})$ is a p-*splitting *if $E^\mathrm{u}(t)$ has dimension p ($\Leftrightarrow$ $E^\mathrm{s}(t)$ has dimension $n - p$) for all $t \in \mathbb{N}$.*

The following theorem is the first main result of this section. It generalizes the dominated splitting feature (2.2) to p-dominant switched linear systems, and also states the converse result, i.e., that any switched linear system admitting a dominated p-splitting is p-dominant.

Theorem 2.11. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$*. The following are equivalent:*

1. SwS *is p-dominant;*

2. *There is* $C \geq 1$, $\mu \in (0,1)$, *and for every switching signal* $\sigma \in \mathcal{S}$, *there is a p-splitting* $(E^{\mathrm{s}}_\sigma, E^{\mathrm{u}}_\sigma)$ *satisfying (i)* $A_{\sigma(t)} E^{\mathrm{s}}_\sigma(t) \subseteq E^{\mathrm{s}}_\sigma(t+1)$ *and* $A_{\sigma(t)} E^{\mathrm{u}}_\sigma(t) = E^{\mathrm{u}}_\sigma(t+1)$ *for all* $t \in \mathbb{N}$, *and (ii) for every* $t_0, t_1 \in \mathbb{N}$, $t_1 \geq t_0$, $x_1 \in E^{\mathrm{s}}_\sigma(t_0)$ *and* $x_2 \in E^{\mathrm{u}}_\sigma(t_0) \setminus \{0\}$,

$$\frac{\|\chi(t_1, t_0, x_1, \sigma)\|}{\|\chi(t_1, t_0, x_2, \sigma)\|} \leq \frac{\|x_1\|}{\|x_2\|} C \mu^{t_1 - t_0}, \tag{2.6}$$

where $\chi(\cdot, \cdot, \cdot, \sigma)$ *is the generator*[5] *of the trajectories of* SwS *with switching signal* σ.

Proof. See Appendix A.2.4. □

The ordered pair $(E^{\mathrm{s}}_\sigma, E^{\mathrm{u}}_\sigma)$ in Item 2 in Theorem 2.11 is called a *dominated p-splitting* for SwS with switching signal σ. The interpretation of (2.6) is that for every trajectory $(\xi, \sigma) : \mathbb{N} \to \mathbb{R}^n \times \Sigma$ of SwS, the component of $\xi(t)$ in $E^{\mathrm{s}}_\sigma(t)$ becomes negligible compared to the component of $\xi(t)$ in $E^{\mathrm{u}}_\sigma(t)$ as $t \to \infty$.

The dominated splitting property is particularly relevant when we look at the system as a switched linear system acting on the *Grassmannian manifold*; that is, instead of looking at the action of the system on points in the state space, we consider its action on subspaces of a fixed dimension in the state space (with an appropriate metric to measure the distance between subspaces). Indeed, in this case, the existence of a dominated p-splitting translates as the *incremental stability* of the system on the Grassmannian manifold, meaning that for every switching signal of the system and for any two p-dimensional subspaces U_1 and U_2 that are close enough to each other, the images of U_1 and U_2 by the system are also p-dimensional subspaces and the distance between them converges exponentially to zero. We refer the reader to Colonius and Kliemann (2014) for a discussion of linear systems acting on the Grassmannian manifold; see also Ghosh and Martin (2002) for the particular case where the subspaces have dimension 1, referred to as dynamical systems in the projective or homogeneous space. See also Rüffer et al. (2013) for the notion of incremental stability.

The incremental stability property is illustrated in Figure 2.7 with the 1-dominant switched linear system of Example 2.2 and with a 2-dominant switched linear system, whose trajectories are radially scaled to the unit sphere:

- The 1-dominant behavior of the first system is captured by the convergence of the normalized trajectories of the system, for any given switching signal $\sigma \in \mathcal{S}$ and for different initial conditions, to two opposite "attracting trajectories" as time goes to ∞.

[5] For a reminder, see Definition 1.36 in Section 1.3.

- The 2-dominant behavior of the second system is captured by the convergence of the normalized trajectories of the system, for any given switching signal $\sigma \in \mathcal{S}$ and for different initial conditions, to a time-varying 2-dimensional subspace as time goes to ∞.

Unlike stable switched linear systems, which all converge to a unique equilibrium, p-dominant switched linear systems allow for richer behaviors.

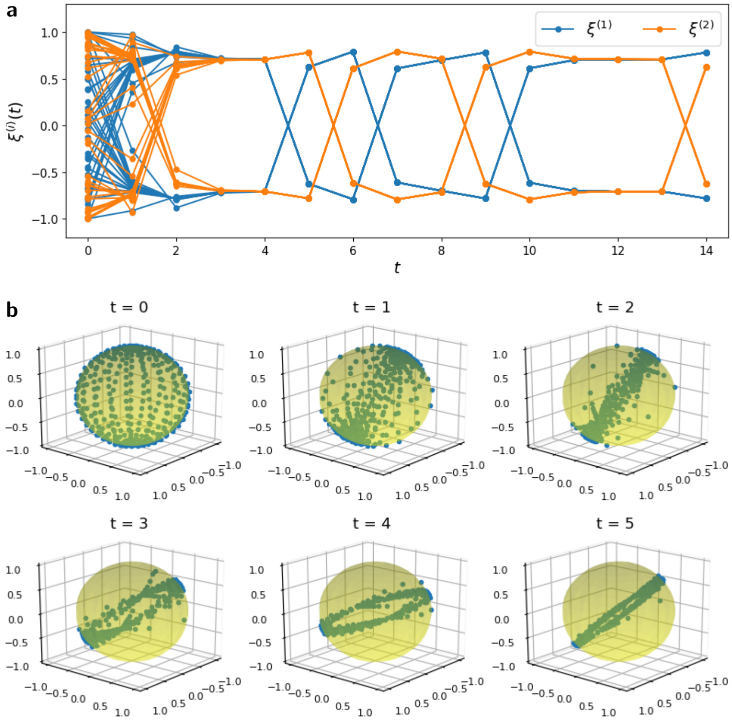


Figure 2.7: **a**: Normalized trajectories of the 1-dominant switched linear system from Example 2.2, starting from different initial conditions and for a random switching signal σ. **b**: Normalized trajectories of a 2-dominant switched linear system starting from different initial conditions and for a random switching signal σ. Each dot represents the projection on the unit sphere of a trajectory $\xi : \mathbb{N} \to \mathbb{R}^3$ with switching signal σ at times $t = 0, 1, \ldots, 5$.

A straightforward consequence of the equivalence of Items 1 and 2 in Theorem 2.11 is that the property of having a dominated invariant splitting $(E^{\mathrm{s}}_\sigma, E^{\mathrm{u}}_\sigma)$ for all $\sigma \in \mathcal{S}$ is *robust to small system perturbations*. The robustness property is instrumental for numerical analysis, and also shows that the property of having a low-dimensional dominant behavior does not occur with probability zero for switched linear systems.

Corollary 2.12. *The property of Item 2 in Theorem 2.11 is robust to small perturbations of the matrices* $\{A_i\}_{i\in\Sigma}$.

Proof. The property of being p-dominant is clearly robust to system perturbations, as for any small enough perturbation of the matrices $\{A_i\}_{i\in\Sigma}$, the dissipation inequalities (2.4) will still be satisfied, with the same automaton, the same set of rates $\{\gamma_\theta\}_{\theta\in\Theta}$ and the same matrices $\{P_q\}_{q\in Q}$. Hence, from the equivalence of Items 1 and 2 in Theorem 2.11, we get the desired result. □

An interesting situation is when the system has a *stable dominated behavior*. This means that the system converges to zero on the dominated component of the splitting, so that the asymptotic behavior of the system is dictated by the dominant component only. In order to characterize switched linear systems with such a property, we first introduce the notion of *cycle-stable* automaton. For that, let us remind the notion of cycle in an automaton.

Definition 2.13. *A* cycle *in an automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$ *is a finite path* $(\theta_t)_{t=0}^{T-1} \subseteq \Theta$ *in* Aut *such that* $\mathsf{t}(\theta_{T-1}) = \mathsf{s}(\theta_0)$ *and for all* $t_0, t_1 \in \{0, \ldots, T-1\}$, $t_0 \neq t_1$, $\mathsf{s}(\theta_{t_0}) \neq \mathsf{s}(\theta_{t_1})$.

A cycle-stable automaton is then defined as an automaton for which all cycles have an average rate product smaller than one (see Definition 2.14 below). This notion appears for instance in the *maximum cycle mean problem* in graph theory; see, e.g., Karp (1978); see also Ahmadi and Parrilo (2012) for applications in switched systems analysis, or Tomar et al. (2020) for applications in quantized control.

Definition 2.14 (Cycle-stable automaton)**.** *Consider an automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$ *and a set of rates* $\{\gamma_\theta\}_{\theta\in\Theta} \subseteq \mathbb{R}_{>0}$. *We say that* Aut *is* cycle-stable *with respect to* $\{\gamma_\theta\}_{\theta\in\Theta}$ *if every cycle* $(\theta_t)_{t=0}^{T-1}$ *in* Aut *satisfies* $\gamma_{\theta_0} \cdots \gamma_{\theta_{T-1}} \leq 1$.

Using the above, we obtain the following characterization of p-dominant switched linear systems with stable dominated behavior.

Theorem 2.15. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$. *The following are equivalent:*

1. SwS *is p-dominant with some automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$ *and some set of rates* $\{\gamma_\theta\}_{\theta\in\Theta} \subseteq \mathbb{R}_{>0}$ *such that* Aut *is cycle-stable with respect to* $\{\gamma_\theta\}_{\theta\in\Theta}$.

2. SwS *satisfies the property of Item 2 in Theorem 2.11 and there is* $D \geq 1$ *and* $\rho \in (0, 1)$ *such that for every switching signal* $\sigma \in \mathcal{S}$, $t_0, t_1 \in \mathbb{N}$, $t_1 \geq t_0$, *and every* $x_1 \in E^{\mathrm{s}}_\sigma(t_0)$,

$$\|\chi(t_1, t_0, x_1, \sigma)\| \leq \|x_1\| D\rho^{t_1 - t_0}.$$

where $\chi(\cdot, \cdot, \cdot, \sigma)$ is the generator of the trajectories of SwS *with switching signal σ.*

Proof. See Appendix A.2.5. □

The property of a stable dominated behavior will be instrumental for instance in the study of p-dominant smooth dynamical systems (see Section 2.3). It will also be useful to perform dimensionality reduction in the computation of the topological entropy of switched linear systems (see Subsection 2.2.4, and Corollary 3.26 in Subsection 3.3.2).

Summarizing, we introduced the concept of path-complete p-dominance for switched linear systems, and we showed that this concept was key for the theoretical analysis of switched linear systems with a low-dimensional dominant behavior, a property made precise thanks to the notion of dominated splitting (see Theorems 2.11 and 2.15). In Subsection 2.2.3, we will address the question of the algorithmic verification of the property of path-complete p-dominance. Before this, in the next subsubsection, we discuss the connections of our approach with other works in the literature.

Discussion and connections with the literature

Our work connects with several other concepts in control and systems theory. For instance, the use of a family of quadratic forms whose decay properties are dictated by an automaton is inspired from *path-complete Lyapunov functions* introduced in the context of stability analysis of switched systems (see, e.g., Ahmadi et al., 2014, Angeli et al., 2017, and Philippe et al., 2019), and from *path-complete positivity* which extends the property of positivity by moving from a single cone to a family of convex cones whose contraction properties are driven by an automaton (see, e.g., Forni et al., 2017). Another important concept in our analysis is the one of *dominated splitting*, which was first introduced in the context of *partial hyperbolicity* and *exponential dichotomy* theory (a generalization of the celebrated works of Smale and Anosov on the horseshoe map; see, e.g., Brin and Pesin, 1974, Hirsch et al., 1977, Pesin, 2004, and Barreira and Valls, 2008). Dominated splittings also received attention in the study of some classes of switched linear systems (see, e.g., Bochi and Gourmelon, 2009, Barreira and Valls, 2009, Avila et al., 2010, and Brundu and Zennaro, 2019). An important tool in these works is the one of invariant *multicone*. In fact, the proof of the converse theorem for p-dominance ($2 \Rightarrow 1$ in Theorem 2.11) is partially grounded in the proof of Bochi and Gourmelon (2009, Theorem B), which shows that a switched linear system under arbitrary switching and with *invertible* matrices admits a dominated splitting for every switching signal if and only if it admits a contracting multicone. Our work extends this result to switched linear systems with constrained switching signal and involving singular matrices and to sets of quadratic p-cones whose contraction properties are driven by an automaton. Another difference with these references is that little attention is given to the algorithmic decidability of the geometric property, whereas our approach is meant to be translated into a practical algorithm for the computation of the quadratic p-cones, as explained in the next subsection.

2.2.3 Algorithmic verification of p-dominance of switched linear systems

In this subsection, we consider the following computational problem: "for a given dominance degree p and a given switched linear system, how can we compute an automaton, a set of rates and a set of symmetric matrices allowing to certify that the system is p-dominant, or conclude that the system is not p-dominant?"

The subsection is organized as follows. First, we describe an algorithm to compute a set of symmetric matrices satisfying the conditions of p-dominance when the automaton and the set of rates are given, or conclude that the system is not p-dominant with this automaton and this set of rates. Then, we address the problem of finding a suitable automaton and a suitable set of rates. Finally, we discuss the use and the complexity of the overall algorithmic framework.

Computation of the symmetric matrices

Consider a switched linear system $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ and an automaton $\mathsf{Aut} = (Q, \Sigma, \Theta)$ accepting every switching signal of SwS. Let $\{\gamma_\theta\}_{\theta\in\Theta}$ be a set of positive rates. Then, according to Definition 2.8, verifying that SwS is p-dominant with Aut and $\{\gamma_\theta\}_{\theta\in\Theta}$ can be addressed by solving the following optimization problem: with variables $\{P_q\}_{q\in Q} \subseteq \mathbb{S}^{n\times n}$ and $\epsilon \in \mathbb{R}$,

$$\max \quad \epsilon \tag{2.7a}$$

$$\text{s.t.} \quad A_{\mathsf{i}(\theta)}^\top P_{\mathsf{t}(\theta)} A_{\mathsf{i}(\theta)} - \gamma_\theta^2 \, P_{\mathsf{s}(\theta)} \preceq -\epsilon I \qquad \forall\, \theta \in \Theta, \tag{2.7b}$$

$$-I \preceq P_q \preceq I \qquad \forall\, q \in Q, \tag{2.7c}$$

$$P_q \in \mathbb{S}_p^{n\times n} \qquad \forall\, q \in Q. \tag{2.7d}$$

The subproblem (2.7a)–(2.7c) is a semidefinite optimization problem. Semidefinite programming has become a standard tool in control theory (see, e.g., Boyd et al., 1994) and many different solvers are available to solve these problems in polynomial time; see, e.g., Nesterov and Nemirovskii (1994), Ben-Tal and Nemirovski (2001) and Boyd and Vandenberghe (2004). On the other hand, the constraints (2.7d) on the inertia of the matrices $\{P_q\}_{q\in Q}$ are not semidefinite constraints (they are actually nonconvex). However, as we will see below, this set of constraints can in fact be dropped without any impact on the outcome of the decision problem "is SwS p-dominant with Aut and $\{\gamma_\theta\}_{\theta\in\Theta}$?". This statement is formalized in Corollary 2.18 below; to simplify its presentation, let us make the following standing assumption on the automaton Aut.

Assumption 2.16. *We assume that* $\mathsf{Aut} = (Q, \Sigma, \Theta)$ *is* essential*, meaning that for every* $q \in Q$ *there is* $(q^+, i^+) \in Q \times \Sigma$ *such that* $(q, i^+, q^+) \in \Theta$ *and there is* $(q^-, i^-) \in Q \times \Sigma$ *such that* $(q^-, i^-, q) \in \Theta$.

Remark 2.4. The notion of essential automaton extends the one of essential graph introduced in the context of abstraction of dynamical systems (see Definition 1.66 in Subsection 1.4.3). Assumption 2.16 can be made without loss of generality (see, e.g., Lind and Marcus, 1995, Proposition 2.2.10), provided that the set of admissible switching signals of the system is *backward shift-invariant*, meaning that for any $\sigma \in \mathcal{S}$ and $t_0 \in \mathbb{N}$, there is $\sigma' \in \mathcal{S}$ such that for all $t \in \mathbb{N}$, $\sigma(t) = \sigma'(t + t_0)$. The shift-invariance property reflects the fact that the set of admissible switching signals does not depend on the specific instant at which the system starts, and thus is generally satisfied in applications.

The following theorem is the second main result of this section. It states that either there is no solution of (2.7a)–(2.7c) with $\epsilon > 0$ and with matrices $\{P_q\}_{q\in Q}$ having the same inertia, or all solutions of (2.7a)–(2.7c) with $\epsilon > 0$ satisfy that all matrices $\{P_q\}_{q\in Q}$ have the same inertia.

Theorem 2.17. *Consider a set of matrices* $\{A_i\}_{i\in\Sigma} \subseteq \mathbb{R}^{n\times n}$*, an automaton* (Q, Σ, Θ) *and a set of rates* $\{\gamma_\theta\}_{\theta\in\Theta} \subseteq \mathbb{R}_{>0}$*. Let Assumption 2.16 hold. Assume there is a feasible solution* $(\{P_q\}_{q\in Q}, \epsilon)$ *of* (2.7b)–(2.7c) *satisfying* $\epsilon > 0$ *and* $\{P_q\}_{q\in Q} \subseteq \mathbb{S}^{n\times n}_k$ *for some* $k \in \{0, \ldots, n\}$*. Then, it holds that* every *feasible solution* $(\{P'_q\}_{q\in Q}, \epsilon')$ *of* (2.7b)–(2.7c) *with* $\epsilon' > 0$ *satisfies that* $\{P'_q\}_{q\in Q} \subseteq \mathbb{S}^{n\times n}_k$.

Proof. See Appendix A.2.6. □

Hence, to verify that SwS is p-dominant with Aut and the rates $\{\gamma_\theta\}_{\theta\in\Theta}$, it suffices to solve the semidefinite optimization problem (2.7a)–(2.7c).

Corollary 2.18. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$*, an automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$ *accepting every* $\sigma \in \mathcal{S}$*, and a set of rates* $\{\gamma_\theta\}_{\theta\in\Theta} \subseteq \mathbb{R}_{>0}$*. Let Assumption 2.16 hold. Then, any optimal solution* $(\{P^\star_q\}, \epsilon^\star)$ *of*

(2.7a)–(2.7c) *satisfies that* $\epsilon^\star > 0$ *and* $\{P_q^\star\}_{q\in Q} \subseteq \mathbb{S}_p^{n\times n}$ if and only if SwS *is* p*-dominant with* Aut *and* $\{\gamma_\theta\}_{\theta\in\Theta}$.

Proof. The "only if" direction is clear: if the optimal solution $(\{P_q^\star\}, \epsilon^\star)$ satisfies the assertions of the corollary, then (2.4) holds with $\{P_q^\star\}_{q\in Q}$ and thus the system is p-dominant with Aut and $\{\gamma_\theta\}_{\theta\in\Theta}$.

The "if" direction is also straightforward: if the system is p-dominant with Aut and $\{\gamma_\theta\}_{\theta\in\Theta}$, then (2.7a)–(2.7c) has a feasible solution with $\epsilon > 0$ and with $\{P_q\}_{q\in Q} \subseteq \mathbb{S}_p^{n\times n}$. It follows that any optimal solution $(\{P_q^\star\}, \epsilon^\star)$ satisfies $\epsilon^\star > 0$, and by Theorem 2.17, it holds that $\{P_q^\star\}_{q\in Q} \subseteq \mathbb{S}_p^{n\times n}$. □

Corollary 2.18 shows that, if the automaton and the set of rates are given, then the verification of p-dominance for a given switched linear system can be reduced to a semidefinite optimization problem, given by (2.7a)–(2.7c), and thus can be solved efficiently (see the after-next subsubsection for a discussion of the complexity). However, nothing is said about the way of finding this automaton and the associated rates. This question is discussed in the next subsubsection.

Constraints on the automaton and the set of rates

The determination of an automaton and a set of rates satisfying the conditions of p-dominance can be challenging in general. We discuss the complexity of finding them and propose heuristics for this problem in the next subsubsection. Before that, in the present subsubsection, we present *necessary conditions* (or *constraints*) that must be satisfied by the automaton and the set of rates so that they can possibly satisfy the conditions of p-dominance. The advantage of these constraints is that they are easy to compute, while their utility is twofold: (i) they can be used to reduce the "search space" for the automaton and the set of rates, by restricting our attention to those that satisfy the constraints, and (ii) they can be used to conclude that a system is not p-dominant if we can show that no automaton or set of rates satisfies the constraints.

The constraints on the automaton and the set of rates are described in the following proposition.

Proposition 2.19. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *and assume that* SwS *is* p*-dominant with an automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$*, a set of rates* $\{\gamma_\theta\}_{\theta\in\Theta} \subseteq \mathbb{R}_{>0}$ *and a set of matrices* $\{P_q\}_{q\in Q} \subseteq \mathbb{S}_p^{n\times n}$*. Let* $(\theta_t)_{t=0}^{T-1}$ *be a cycle in* Aut*, and let* $\bar{A} = A_{\mathrm{i}(\theta_{T-1})} \cdots A_{\mathrm{i}(\theta_0)}$ *and* $\bar{\gamma} = \gamma_{\theta_0} \cdots \gamma_{\theta_{T-1}}$*. Then, it holds that*

1. *The matrix* $\bar{A}$ *has* p *eigenvalues with modulus* $|\lambda_i| > \bar{\gamma}$ *and* $n-p$ *eigenvalues with modulus* $|\lambda_i| < \bar{\gamma}$*;*

2. *The eigenspace associated to the p eigenvalues of $\bar{A}$ with modulus $|\lambda_i| > \bar{\gamma}$ is contained in $\mathcal{K}(P_{\mathsf{s}(\theta_0)})$, and the eigenspace associated to the $n-p$ eigenvalues of $\bar{A}$ with modulus $|\lambda_i| < \bar{\gamma}$ is contained in $\mathbb{R}^n \setminus \operatorname{int} \mathcal{K}(P_{\mathsf{s}(\theta_0)})$.*

Proof. These constraints follow in fact from the observation that any cycle in Aut defines a p-dominant LTI system (see also Proposition 2.3 in Subsection 2.2.1). See Appendix A.2.7 for details. □

Condition 1 above is particularly useful to reduce the search space for the rates if the automaton is given. Condition 2 is useful to exclude automata that cannot satisfy the dissipation inequalities (2.4) *for any set of rates.*

This is illustrated in the two examples below. In particular, Example 2.4 explains how the rates were selected in Example 2.2, and Example 2.5 shows that the switched linear system of Example 2.2 cannot be 1-dominant with respect to a single quadratic 1-cone, i.e., with an automaton with a single node.

Example 2.4. In Example 2.2, we used the set of rates $\{\gamma_\theta\}_{\theta\in\Theta}$ defined by $\gamma_\theta = 0.32$ for all $\theta \in \Theta$ to show that SwS is 1-dominant with the automaton Aut_2 in Figure 2.5. These values of the rates were somehow the most natural choice regarding the constraints obtained from Condition 1 in Proposition 2.19 when p is fixed to 1:

- The rate $\gamma_{\mathtt{a1a}}$, associated to the loop $\mathtt{a} \xrightarrow{1} \mathtt{a}$, must satisfy $\lambda_1(A_1) = 1 > \gamma_{\mathtt{a1a}} > \lambda_2(A_1) = 0.1$. In the example, we have used the geometric mean of the bounds: $\gamma_{\mathtt{a1a}} = \tilde{\gamma} := \sqrt{0.1}$. Similarly, we have used $\gamma_{\mathtt{b2b}} = \tilde{\gamma}$ for the rate associated to $\mathtt{b} \xrightarrow{2} \mathtt{b}$.

- By looking at the cycle $\mathtt{a} \xrightarrow{2} \mathtt{b} \xrightarrow{1} \mathtt{a}$, we get that the associated rates must satisfy $|\lambda_1(A_1A_2)| \approx 0.6 > \gamma_{\mathtt{a2b}}\gamma_{\mathtt{b2a}} > |\lambda_2(A_1A_2)| \approx 0.017$. In the example, we have used $\gamma_{\mathtt{a2b}} = \gamma_{\mathtt{b1a}} = \sqrt[4]{0.6 \cdot 0.017}$ (which in this example can be shown to be equal to $\tilde{\gamma}$).

See also Figure 2.8-a for a representation of the above quantities. Note that these rates are not the only ones satisfying the constraints of Proposition 2.19 and that the 1-dominance of the system with this automaton and this set of rates was not guaranteed a priori, but it happened to be the case for this example.

When α increases, the eigenvalues of A_1A_2 (and A_2A_1) get closer to each other; see Figure 2.8-c. For $\alpha < 3-2\sqrt{2} \approx 0.1716$, SwS is still 1-dominant with the same automaton as above and with the rates chosen in the same way as above. However, the contraction property (2.5) gets more "fragile", in the sense that the images of $\mathcal{K}(P_{\mathtt{a}})$ and $\mathcal{K}(P_{\mathtt{b}})$ get closer to the boundary of the cones; see Figure 2.8-d. In fact, when $\alpha \geq 3-2\sqrt{2}$, SwS is not path-complete 1-dominant anymore since the matrix $A_1 A_2$ has two complex conjugated eigenvalues (hence with the same modulus), and thus it does not satisfy Condition 1 in Proposition 2.19, *for any set of rates.*

Example 2.5. From Condition 2 in Proposition 2.19, it follows that the system of Example 2.2 cannot be 1-dominant with the automaton Aut_1 in Figure 2.5 (for any set of rates). Indeed, if it was the case, then the cone $\mathcal{K}(P_{\mathtt{a}})$ would contain the dominant eigenvectors of A_1 and A_2. Because $\mathcal{K}(P_{\mathtt{a}})$ consists of two convex components, this would imply that $\mathcal{K}(P_{\mathtt{a}})$ also contains the eigenvectors associated to $\lambda_2 = \alpha$ of A_1 or A_2 (indeed, one can readily check in Figure 2.8-b that any quadratic 1-cone containing the two dominant eigenspaces (solid lines) will also contain one of the dominated eigenspaces (dashed lines)), a contradiction with Condition 2 in Proposition 2.19.

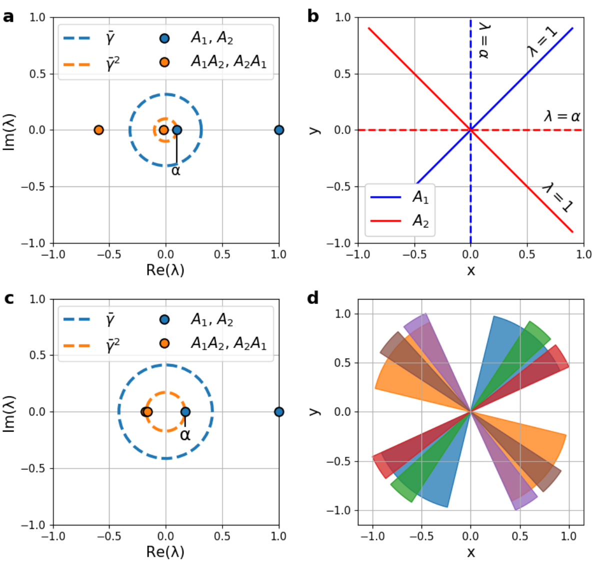


Figure 2.8: **a**: Eigenvalues of A_1, A_2, A_1A_2 and A_2A_1 (see Example 2.2). **b**: Eigenvectors of A_1 and A_2, associated to $\lambda_1 = 1$ and $\lambda_2 = \alpha$. **c–d**: A_1 and A_2 are as in Example 2.2 with $\alpha = 0.1715$. **c**: Eigenvalues of A_1, A_2, A_1A_2 and A_2A_1. **d**: Quadratic 1-cones $\mathcal{K}(P_{\mathtt{a}})$ and $\mathcal{K}(P_{\mathtt{b}})$ and their images by A_1 and A_2 (the color code is the same as in Figure 2.6).

Complexity and comparison with the literature

For a given automaton and a given set of rates, the verification of p-dominance with this automaton and this set of rates can be computed efficiently by using Corollary 2.18 and interior-point algorithms to solve the associated semidefinite optimization problem; the complexity is in $\mathcal{O}(|Q|^2|\Theta|^{1.5}n^{6.5})$, where $|Q|$ and $|\Theta|$ are the number of nodes and the number of transitions in the automaton, and n is the dimension of the system (see, e.g., Ben-Tal and Nemirovski, 2001, Section 6.6.3).

On the other hand, finding a suitable automaton, or concluding that none exists, can be challenging in general. The following systematic approach for finding a suitable automaton (if one exists) can be used:

1. Consider an automaton capturing the sequences of modes of length T of the system (using for instance *De Bruijn automata*; see, e.g., Gross et al., 2014);

2. For such an automaton, try to find a set of rates and symmetric matrices satisfying the conditions of p-dominance (using for instance Bilinear Matrix Inequalities solvers);

3. If no such set of rates and symmetric matrices can be found, then increase the value of T and go back to step 1.

The proof of $2 \Rightarrow 1$ in Theorem 2.11 (see Appendix A.2.4) ensures that if the system is p-dominant, then the above approach will always find an automaton and a set of rates with which the system is p-dominant. However, this approach is *highly inefficient*; the reason is that the size of the automata constructed in this way increases very rapidly in general, and Bilinear Matrix Inequalities solvers are known to scale very poorly with the dimension of the problem (which depends on the size of the automaton). Consequently, this approach was not used in the numerical examples we considered (see Subsection 2.2.4), due to its lack of practical applicability. Another comment is also in order: the above approach is guaranteed (thanks to Theorem 2.11) to find a suitable automaton and a suitable set of rates if the system is p-dominant, however there is no criterion for deciding when to stop to looking for an automaton, so that if the system is not p-dominant, then the above procedure will never stop. The above algorithm is thus *semi-complete*, meaning that it finds a suitable automaton in finite time for systems that are p-dominant, but never terminates for systems that are not p-dominant. If one wants to show that a system is *not* p-dominant, they may use the constraints described in the previous subsubsection, but those are not guaranteed in general to be able to show that the system is not p-dominant.

Remark 2.5. Heuristics and more sophisticated optimization approaches can be used to search for a suitable automaton for p-dominance verification. We think for instance to *Counter-Example Guided Abstraction Refinement* (CEGAR) techniques, which are used for instance to find Lyapunov functions and automata for the stability analysis of switched linear systems; see e.g., Mitra (2021, Section 8.7) and the references therein. In a nutshell, the idea of CEGAR is to used a candidate automaton, set of rates and/or set of matrices, and to check whether p-dominance can be proved with this automaton/set of rates/set of matrices. If this is not the case, there should be a witness (called a *counter-example*) of the infeasibility. The idea is then to used this counter-example to improve the automaton/set of rates/set of matrices in order to be able to show that the system is p-dominant.

These rather deceptive results must be contrasted with the following two observations. The first one is that the problem of p-dominance verification is a difficult problem in itself, as it supersedes the problem of stability of switched linear systems (corresponding to 0-dominance with rates smaller than one), which is known to be undecidable (see, e.g., Jungers, 2009, Section 2.2). Thus, one may not hope to have a complete, let alone efficient, algorithm for the verification of p-dominance of switched linear systems in general.

The second one is that, despite these negative theoretical results, it appears that in many practical situations, a suitable automaton can be easily guessed from the structure of the problem and from Condition 2 in Proposition 2.19. Similarly, the search space for the rates can be considerably reduced by using the symmetry of the problem (present in many applications) and Condition 1 in Proposition 2.19. As a consequence, finding the automaton and the rates was *not a serious limitation* in the various numerical examples presented in the thesis.

Finally, for some specific applications, the rates and the automaton can be assumed to be fixed or can be obtained in an easier way. This is the case for instance for the study of the property of hyperbolicity for smooth dynamical systems (see Subsection 2.3.3). Indeed, for this application, the rates can be assumed to be equal to one and the automaton can be obtained by computing abstractions of the system (the latter is not necessarily easy—especially when the dimension grows—, but at least it has the advantage of having several toolboxes available for this problem). Another situation where finding the automaton and the set of rates can be simplified is for applications in quantized control. Indeed, if one wants to obtain upper bounds on the topological entropy of a switched linear system, then at the cost of possibly adding conservatism to the obtained bounds, they can fix the automaton and assume that the rates are uniform (i.e., all have the same value). Indeed, by tuning the common value of the rates and using the algorithm for the verification of p-dominance when the automaton and the rates are given, one can find values of p for which the system is p-dominant with this automaton and these rates, and from that, derive bounds on the topological entropy of the system; see the second subsubsection of Subsection 2.2.4 and the second subsubsection of Subsection 2.3.3 for details on how these bounds are obtained.

In the next subsection, we present numerical examples and examples of applications of the theory of p-dominance and the associated algorithmic framework. Before that, we would like to briefly review the literature on computational questions related to dominance analysis.

The verification of the property of having a dominated 1-splitting has been studied by Forni et al. (2017) and Brundu and Zennaro (2019). The first one presents an algorithm for constructing a *common* convex cone that is contracted by the system. However, the restriction to a single common cone adds conservatism to the approach, so that it is not possible to capture every switched linear system with a dominated 1-splitting. The second one describes an algorithm for computing an invariant multicone for switched linear systems, with invertible matrices, that have a dominated 1-splitting. However, the computed multicone is not strictly invariant, so that it does not allow to deduce that the system admits a dominated 1-splitting.[6] Closer to our work, the question of algorithmic verification of the property of p-dominance (with general p) was addressed by Forni and Sepulchre (2019), for continuous-time dynamical systems that have a *common* quadratic p-cone contracted by the prolonged system. The concept of path-complete p-dominance introduced above extends this property to discrete-time switched linear systems and to *families* of quadratic p-cones whose contraction properties are dictated by an automaton.

2.2.4 Numerical examples and applications

In this subsection, we present some examples of application of the theory of path-complete p-dominance for switched linear systems developed in the previous subsections. Further examples, involving nonlinear dynamical systems, will be presented in Subsection 2.3.3 after we presented how this theory can be used for the analysis of "p-dominant nonlinear dynamical systems" (see Section 2.3).

[6] Let us also mention that the approach used in Forni et al. (2017) and Brundu and Zennaro (2019) for the computation of the cone/multicone—which relies on polyhedral set methods, thriving on the fact that the involved sets can be described as the finite union of disjoint convex polyhedral cones—is hardly generalizable to the verification of p-dominance with $p \geq 2$. Indeed, cones that are compatible with p-dimensional dominant behavior are in general not representable as the finite union of convex cones.

1-dominance and population dynamics

Consider the switched linear system $\mathsf{SwS} \sim (\mathbb{R}^6, \{A_i\}_{i\in\Sigma})$ under arbitrary switching, with $\Sigma = \{1, 2\}$ and

$$A_1 = \begin{bmatrix} 0.1 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0.95 & 0 & 0 & 0.27 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0.255 & 0 \\ 0 & 0 & 0 & 0.21 & 0.63 & 0.49 \\ 0 & 0 & 0 & 0.63 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.595 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.07 & 0.14 & 0.14 & 0 & 0 & 0 \\ 0.665 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.63 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.9 & 0.7 \\ 0.285 & 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0.27 & 0 & 0 & 0.85 & 0 \end{bmatrix}.$$

This system, adapted from Schmidbauer et al. (2012, Eq. 4), may appear for instance in the study of *aged-structured populations* with migration between the populations. In this example, the 1^{st}, 2^{nd} and 3^{rd} (resp. 4^{th}, 5^{th} and 6^{th}) components of the trajectories of the system, represent the number of individuals in each of the three age classes of some urban (resp. rural) population. Each population evolves according to the *Leslie model* (see, e.g., Farina and Rinaldi, 2000, Chapter 13, and Schmidbauer et al., 2012), and there is migration either from villages to cities (A_1) or from cities to villages (A_2).[7]

A central question in the study of population dynamics is whether the asymptotic *composition* of the population depends on the initial condition; see, e.g., Farina and Rinaldi (2000), Golubitsky et al. (1975), Tuljapurkar (1982), Schmidbauer and Rösch (1995) or Schmidbauer et al. (2012). The population composition at time $t \in \mathbb{N}$ is represented by the normalized vector $\xi(t)/\|\xi(t)\|_1$, where $\xi : \mathbb{N} \to \mathbb{R}^6$ is the trajectory of SwS for some switching signal σ, and $\|\cdot\|_1$ is the 1-norm. Using dominance analysis, we will show that for any switching signal $\sigma \in \mathcal{S}$, the population composition is ultimately independent of its initial condition, meaning that for any two $x_1, x_2 \in (\mathbb{R}_{\geq 0})^6 \setminus \{0\}$,

$$\lim_{t\to\infty} \left| \frac{\chi(t, x_1, \sigma)}{\|\chi(t, x_1, \sigma)\|_1} - \frac{\chi(t, x_2, \sigma)}{\|\chi(t, x_2, \sigma)\|_1} \right| = 0.$$

[7] This is where our model differs from Schmidbauer et al. (2012): instead of having a single matrix that encodes at the same time the migrations from villages to cities and from cities to villages, we have decomposed this matrix in two matrices, A_1 and A_2, to get a switched linear system, and we assume that system switches between the two modes (e.g., as a consequence of external factors, like climate, conjuncture, etc.).

To do this, we consider the automaton Aut_1 in Figure 2.5 (which accepts every $\sigma \in \mathcal{S}$), together with the set of rates $\gamma_{\mathtt{a1a}} = 0.79$ and $\gamma_{\mathtt{a2a}} = 0.95$ (these rates were selected with the same technique as in Example 2.4). Using the algorithm described in Subsection 2.2.3, we can show that the system is 1-dominant with this automaton and this set of rates. Thus, from Theorem 2.11, we may conclude that the normalized trajectories of the system converge to the same normalized trajectory. In other words, for any switching signal, the asymptotic composition of the population is independent of its initial condition; this is illustrated in Figure 2.9, where a random switching signal was chosen, and we observe that the trajectories, starting from different initial conditions, have ultimately the same population composition.

Remark 2.6. Note that the automaton Aut_1 and the above set of rates are not the only ones satisfying the constraints (2.4) of p-dominance and, even if the system has a 1-dimensional asymptotic behavior, it was not guaranteed a priori that the system is 1-dominant with this automaton and this set of rates. If it had not been the case, then one would have needed to search for more complex automata (such as Aut_2 or Aut_3 in Figure 2.5 for instance). This would have increased the complexity of the problem, but not the conclusion on the asymptotic behavior of the system.

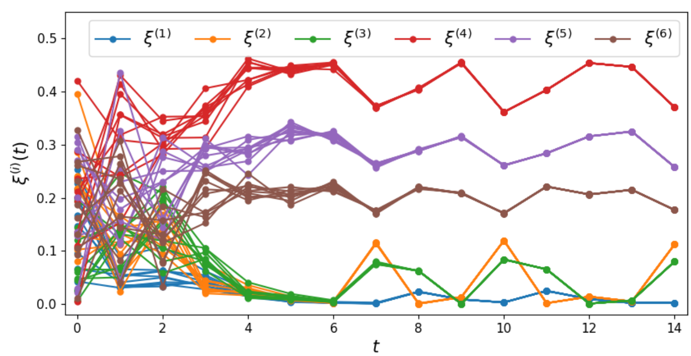


Figure 2.9: Normalized trajectories of SwS, starting from different initial conditions and for a random switching signal $\sigma \in \mathcal{S}$. The trajectories are normalized such that $\sum_i \xi^{(i)}(t) = 1$. We observe that all normalized trajectories converge to the same normalized trajectory as t goes to ∞.

The property that the normalized trajectories, for a same switching signal, converge to each other independently of their initial condition is known as the property of incremental stability of the normalized system $\xi^+ = A_\sigma \xi / \|A_\sigma \xi\|$ (see also the discussion below Theorem 2.11). Different approaches have been proposed in the literature to verify the incremental stability property of dynamical and switched systems; see, e.g., Lohmiller and Slotine (1998), Angeli (2002) and Forni and Sepulchre (2014). These approaches, also called *contraction analysis*, generally amount to construct a Lyapunov function for the prolonged system (see Definition 1.29 in Subsection 1.2.3), which describes the evolution of the sensitivity of the system to the initial condition (see Proposition 1.30 in Subsection 1.2.3). For the above system, the sensitivity matrix, denoted here by $\delta\xi(t)$, of the trajectory at time t with respect to the initial condition satisfies $\delta\xi^+ = \big[\frac{A_\sigma}{\|A_\sigma\xi\|} - \frac{(A_\sigma\xi)(A_\sigma\xi)^\top}{\|A_\sigma\xi\|^3}\big]\delta\xi$. It thus amounts to find a Lyapunov function for a linear system whose transition matrix depends on x (the state of the trajectory), so that there is an infinite number of such matrices. By contrast, the approach proposed above, relying on dominance analysis, uses the switched linear system as it is, so that the number of matrices is finite.

p-dominance and bounds on the topological entropy

In Subsection 1.5.2, we introduced the concept of topological entropy for hybrid systems. In Chapter 3, we will see that this quantity is instrumental for the study of networked switched linear systems. In this subsubsection, we show that the theory of p-dominance can be used to derive bounds on the topological entropy of switched linear systems.

To illustrate this, consider the switched linear system $\mathsf{SwS} \sim (\mathbb{R}^3, \{A_i\}_{i\in\Sigma})$ under arbitrary switching, with $\Sigma = \{1,2,3\}$,

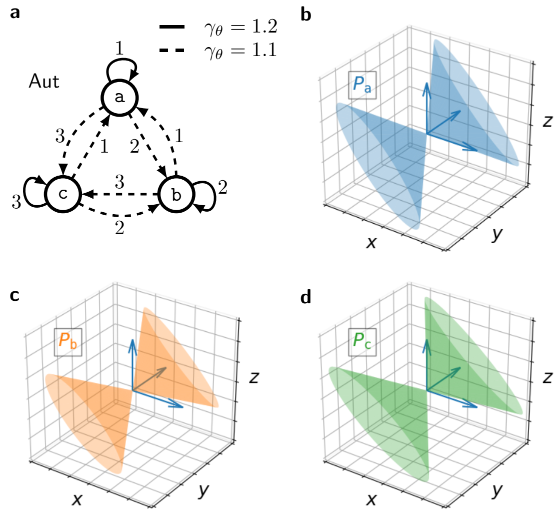
$$A_1 = \begin{bmatrix} 2 & -0.3 & -0.3 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.7 & 0 & 0 \\ -0.3 & 2 & 0.3 \\ 0 & 0 & 0.7 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0.7 & 0 \\ -0.3 & 0.3 & 2 \end{bmatrix}.$$

This system can be shown to be 1-dominant with the automaton $\mathsf{Aut} = (Q, \Sigma, \Theta)$ and the set of rates $\{\gamma_\theta\}_{\theta\in\Theta}$ depicted in Figure 2.10-a. The automaton was obtained by using the symmetry of the system,[8] and the set of rates was derived by making several guesses and using Condition 1 in Proposition 2.19 to reduce the "search space" for the rates. The 1-dominance of SwS with this automaton and set of rates was verified thanks to Corollary 2.18; for the interested reader, the associated quadratic 1-cones are depicted in Figure 2.10-b,c,d.

Let $\sigma : \mathbb{N} \to \Sigma$ be a switching signal for SwS and let $(\theta_t)_{t=0}^\infty \subseteq \Theta$ be a path in Aut such that $\sigma(t) = \mathsf{i}(\theta_t)$ for all $t \in \mathbb{N}$. We will see in Subsection 3.2.3 (Theorem 3.11) that the *topological entropy*, denoted by $h_{\text{top}}(\mathsf{SwS};\sigma)$, of

[8] However, by using Condition 2 in Proposition 2.19, it can be shown that SwS is not 1-dominant with the trivial automaton (that is, the automaton with only one node), for any set of rates. SwS is thus another example of switched linear system under arbitrary switching that requires a non-trivial automaton to be 1-dominant (see also Example 2.5).

Figure 2.10: **a**: Automaton and set of rates for which SwS is 1-dominant. **b–d**: Quadratic 1-cones associated to the matrices $\{P_q\}_{q \in Q} \subseteq \mathbb{S}_1^{n \times n}$ computed thanks to Corollary 2.18 to show 1-dominance of SwS with the automaton and the set of rates in **a**.



the *linear time-varying system* associated to SwS with switching signal σ is bounded by

$$\limsup_{T\to\infty} \frac{1}{T} \max\{0, \log_2(\gamma_{\theta_0} \cdots \gamma_{\theta_{T-1}})\} \leq h_{\text{top}}(\mathsf{SwS}; \sigma) \leq$$
$$2 \limsup_{T\to\infty} \frac{1}{T} \max\{0, \log_2(\gamma_{\theta_0} \cdots \gamma_{\theta_{T-1}})\} + \log_2(e)\, \hat{\varrho}(\mathsf{SwS}),$$

where $\hat{\varrho}(\mathsf{SwS})$ is the joint spectral radius of SwS (see Definition 1.50 in Subsection 1.3.2). This gives the following interval for the value of $h_{\text{top}}(\mathsf{SwS}; \sigma)$:

$$h_{\text{top}}(\mathsf{SwS}; \sigma) \in [\log_2(1.1), 2\log_2(1.2) + \log_2(2)] \approx [0.138, 1.526].$$

These bounds can be compared for instance with the bounds in Yang et al. (2020, Theorem 1) which provide $h_{\text{top}}(\mathsf{SwS}; \sigma) \in [0, 3]$.

2.3 Dominance analysis of smooth dynamical systems

In the previous section, we introduced the concept of p-dominance for discrete-time switched linear systems. In this section, we leverage this approach for the study of discrete-time smooth dynamical systems whose linearized dynamics can be decomposed into a p-dimensional dominant component and a complementary $(n-p)$ dominated component (with n the dimension of the system). Thriving on the algorithmic framework for the analysis of p-dominance of discrete-time switched linear systems and on the technique of abstraction of dynamical systems, we provide an algorithmic framework for the study of

p-dominance for discrete-time smooth dynamical systems. Finally, we discuss two applications of the theory of p-dominance for discrete-time smooth dynamical systems, namely regarding the verification of the property of hyperbolicity and the estimation of the topological entropy of these systems.

The section is organized as follows. In Subsection 2.3.1, we introduce the notion of p-dominance for discrete-time smooth dynamical systems. In Subsection 2.3.2, we present the algorithmic framework for the study of the property of p-dominance for these systems. Finally, in Subsection 2.3.3, we discuss some applications, supported by numerical examples, of the above theory and algorithmic framework.

Notation. In this section, all considered dynamical systems are discrete-time smooth dynamical systems, and thus for the sake of brievety, we will refer to them simply as *dynamical systems*. If A and B are sets, we write $A \Subset B$ if $A \subseteq 2^B$. The Minkowski sum of $A \subseteq \mathbb{R}^d$ and $B \subseteq \mathbb{R}^d$ (or $\{x\} \subseteq \mathbb{R}^d$) is denoted by $A + B$ (or $A + x$), and the convex hull of A is denoted by $\operatorname{conv} A$. The set of symmetric matrices in $\mathbb{R}^{n\times n}$ is denoted by $\mathbb{S}^{n\times n}$. For $P, Q \in \mathbb{S}^{n\times n}$, we write $P \succ Q$ ($P \succeq Q$) if $P - Q$ is positive (semi)definite.

2.3.1 p-dominant smooth dynamical systems

The notion of p-dominant dynamical system draws on the one of path-complete p-dominant switched linear system, introduced in the previous section. For a reminder, the trajectories $\xi : \mathbb{N} \to \mathbb{R}^n$ of a dynamical system $\mathsf{Sys} = (\mathbb{R}^n, f)$ satisfy $\xi(t+1) = f(\xi(t))$ for all $t \in \mathbb{N}$; the generator[9] of Sys is denoted by $\chi(\cdot,\cdot;\mathsf{Sys})$ (or χ if Sys is clear from the context); and the derivative of f is denoted by $\frac{\partial f}{\partial x}$. The sensitivity of the trajectories to the initial condition, defined by $\frac{\partial \chi}{\partial x}$, satisfies the linear system

$$\frac{\partial \chi}{\partial x}(t+1, x) = \frac{\partial f}{\partial x}(\chi(t,x))\frac{\partial \chi}{\partial x}(t,x) \qquad \text{for all } t \in \mathbb{N} \text{ and } x \in \mathbb{R}^n$$

(see Proposition 1.30 in Subsection 1.2.3). The system describing the evolution of $\frac{\partial \chi}{\partial x}$ can thus be seen as a switched linear system with an infinite number of matrices, given by $\{\frac{\partial f}{\partial x}(x)\}_{x\in\mathbb{R}^n}$, and with switching signal given by the trajectory of the system; in other words, "$A_{\sigma(t)}$" is replaced by "$\frac{\partial f}{\partial x}(\chi(t,x))$". The idea is to apply the tools developed within the framework of dominance analysis for switched linear systems, to study the existence of dominated splittings for the linearized dynamics of Sys, that is, for the dynamics of the sensitivity matrix $\frac{\partial \chi}{\partial x}$.

Since we are now considering switched linear systems with an infinite number of matrices, we resort to automata whose transitions are associated to sets

[9] For a reminder, see Definition 1.20 in Subsection 1.2.1.

of matrices instead of single matrices. Therefore, we define the notion of admissible sequences of matrices with respect to an automaton and an associated set of sets of matrices.

Definition 2.20 (Admissible sequence of matrices for an automaton and sets of matrices)**.** *Consider an automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$*, and for each* $i \in \Sigma$*, let* $\mathcal{A}_i \subseteq \mathbb{R}^{n\times n}$*. A function* $\hat{A} : \mathbb{N} \to \mathbb{R}^{n\times n}$ *is said to be* admissible *for* Aut *and* $\{\mathcal{A}_i\}_{i\in\Sigma}$ *(or* accepted *by* Aut *and* $\{\mathcal{A}_i\}_{i\in\Sigma}$*) if there is a path* $(\theta_t)_{t=0}^{\infty}$ *in* Aut *satisfying that for all* $t \in \mathbb{N}$*,* $\hat{A}(t) \in \mathcal{A}_{\mathsf{i}(\theta_t)}$*.*

This allows us to define the notion of p-dominant dynamical system. As in Section 2.2, p is a fixed integer between 0 and n, called the *degree of dominance.* Let us remind that a matrix $P \in \mathbb{S}^{n\times n}$ is said to have *inertia* $(k, 0, n-k)$ if it has k negative eigenvalues and $n-k$ positive eigenvalues. The set of matrices of $\mathbb{S}^{n\times n}$ with inertia $(k, 0, n-k)$ is denoted by $\mathbb{S}_k^{n\times n}$ (see Definition 2.1 in Subsection 2.2.1).

Definition 2.21 (p-dominant dynamical system)**.** *Consider a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$ *and a set* $\Lambda \subseteq \mathbb{R}^n$ *forward invariant for* Sys*. We say that* Sys *is* p-dominant *on* Λ *if there is an automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$*, a set of sets of matrices* $\{\mathcal{A}_i\}_{i\in\Sigma} \Subset \mathbb{R}^{n\times n}$*, a set of rates* $\{\gamma_\theta\}_{\theta\in\Theta} \subseteq \mathbb{R}_{>0}$ *and a set of matrices* $\{P_q\}_{q\in Q} \subseteq \mathbb{S}_p^{n\times n}$ *such that*

1. *for every* $x \in \Lambda$*, the matrix sequence* $\hat{A}_x : \mathbb{N} \to \mathbb{R}^{n\times n}$*, defined by* $\hat{A}_x(t) = \frac{\partial f}{\partial x}(\chi(t, x))$*, is admissible for* Aut *and* $\{\mathcal{A}_i\}_{i\in\Sigma}$*;*
2. *for every* $\theta \in \Theta$ *and* $A \in \mathcal{A}_{\mathsf{i}(\theta)}$*,*

$$A^{\mathsf{T}} P_{\mathsf{t}(\theta)} A - \gamma_\theta^2 P_{\mathsf{s}(\theta)} \prec 0. \tag{2.8}$$

We present below a numerical example of a 2-dominant dynamical system to illustrate the above concepts.

Example 2.6 (2-dominant dynamical system)*.* Consider the nonlinear dynamical system[10] $\mathsf{Sys} = (\mathbb{R}^3, f)$ defined by

$$\begin{cases} \xi^{(1)}(t+1) = \xi^{(1)}(t) + 0.3\xi^{(2)}(t), \\ \xi^{(2)}(t+1) = 0.3\sin(\xi^{(1)}(t)) - 0.15\xi^{(1)}(t) + 0.7\xi^{(2)}(t) + 0.03\xi^{(3)}(t), \\ \xi^{(3)}(t+1) = -1.5\xi^{(1)}(t) + 0.925\xi^{(3)}(t). \end{cases}$$

[10] This system is a discrete-time version of the Duffing oscillator actuated by a DC motor; adapted from Forni and Sepulchre (2019, Subsection 4.C).

We will show that Sys is 2-dominant on $\mathbb{R}^3$. Therefore, note that the derivative of f at $x \in \mathbb{R}^n$, which satisfies

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} 1.0 & 0.3 & 0.0 \\ 0.3\cos(x^{(1)}) - 0.15 & 0.7 & 0.03 \\ -1.5 & 0.0 & 0.925 \end{bmatrix},$$

depends on x only via $\cos(x^{(1)})$. Thus, we consider the following four sets of matrices:

$$\mathcal{A}_i = \left\{\tfrac{\partial f}{\partial x}(x) : x \in \mathbb{R}^3 \text{and} \cos(x^{(1)}) \in I_i\right\} \subseteq \mathbb{R}^{3\times 3}, \qquad \text{for each } i \in \Sigma \doteq \{1,2,3,4\},$$

where $I_1 = [-1, -\frac{1}{2}]$, $I_2 = [-\frac{1}{2}, 0]$, $I_3 = [0, \frac{1}{2}]$, $I_4 = [\frac{1}{2}, 1]$. By construction, it holds that for every $x \in \mathbb{R}^3$, $\frac{\partial f}{\partial x}(x) \in \bigcup_{i\in\Sigma} \mathcal{A}_i$. Now, consider the automaton Aut $= (Q, \Sigma, \Theta)$ depicted in Figure 2.11-a. It is readily checked that this automaton accepts every switching signal $\sigma : \mathbb{N} \to \Sigma$. Thus, it holds that any matrix sequence $\hat{A} : \mathbb{N} \to \bigcup_{i\in\Sigma} \mathcal{A}_i$ is admissible for Aut and $\{\mathcal{A}_i\}_{i\in\Sigma}$, so that Aut and $\{\mathcal{A}_i\}_{i\in\Sigma}$ satisfy Condition 1 in the definition of p-dominance for Sys (Definition 2.21).

To show that Sys is 2-dominant with Aut and $\{\mathcal{A}_i\}_{i\in\Sigma}$, we use the set of rates $\{\gamma_\theta\}_{\theta\in\Theta}$ given by $\gamma_\theta = \tilde{\gamma} := 0.83$ for all $\theta \in \Theta$. This set of rates was derived by making several guesses and using a condition similar to the one in Proposition 2.19 (see Proposition 2.26 in the next subsection); see also Figure 2.11-b for an illustration. It can be shown that Condition 2 in Definition 2.21 holds with Aut, $\{\mathcal{A}_i\}_{i\in\Sigma}$, the above set of rates $\{\gamma_\theta\}_{\theta\in\Theta}$ and some set of matrices $\{P_q\}_{q\in Q} \subseteq \mathbb{S}_2^{3\times 3}$. In other words the system is 2-dominant. For the interested reader, let us mention that the set of matrices $\{P_q\}_{q\in Q}$ was computed using the algorithm described in the next subsection (see Theorems 2.23 and 2.24); the quadratic 2-cones associated to the matrices $\{P_q\}_{q\in Q}$ are represented in Figure 2.11-c.

Example 2.7 (Example 2.6 continued)*.* The 2-dominance of Sys in Example 2.6 translates by the fact that the asymptotic behavior of Sys is at most 2-dimensional, in the sense that the *ω-limit set* (see Definition 1.24 in Subsection 1.2.1) of any bounded trajectory of Sys is contained in a 2-dimensional manifold. In fact, this follows from the following observation.

Proposition 2.22*. With* Sys *and* $\{P_q\}_{q\in Q}$ *as above, if* x_1, x_2 *are two points in the* ω*-limit set of some bounded trajectory* $\xi : \mathbb{N} \to \mathbb{R}^3$ *of* Sys*, then* $x_1 - x_2$ *belongs to* $\bigcup_{q\in Q} \mathcal{K}(P_q)$*.*

Proof. The proof relies on the fact that the automaton is cycle-stable (see Definition 2.14 in Subsection 2.2.2) and on the fact that there is a common 2-

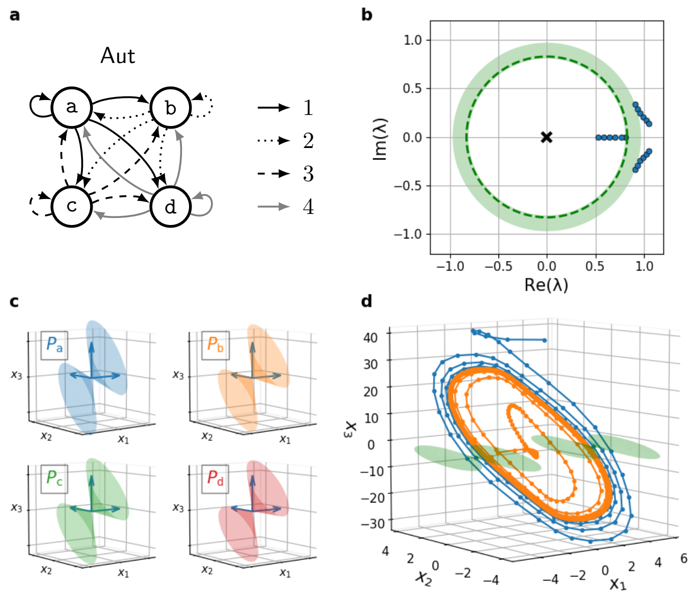


Figure 2.11: **a**: Automaton used to analyze the 2-dominance of Sys (the transitions are labeled according to the legend on the right). **b**: The blue dots represent the eigenvalues of $\frac{\partial f}{\partial x}(x)$ for 6 randomly selected points $x \in \mathbb{R}^3$. By Proposition 2.26, the value of $\tilde{\gamma}$ must lie in the green strip. The dashed circle corresponds to the value $\tilde{\gamma} = 0.83$. **c**: Quadratic 2-cones associated to the matrices $\{P_q\}_{q \in Q}$. **d**: Two trajectories of Sys (in blue and orange). The quadratic 2-cone associated to P_{c} (in green) centered at 2 different points of the ω-limit set of the trajectory in blue does not intersect this ω-limit set, as predicted by Proposition 2.22.

dimensional subspace included in $\mathcal{K}(P_q)$ for all $q \in Q$. See Berger and Jungers (2021b, Proposition 17) for details. □

To illustrate the above, we have represented in Figure 2.11-d two trajectories of Sys, starting from different initial conditions. We verify that the ω-limit set of each trajectory is at most 2-dimensional.

Remark 2.7. The result presented in Example 2.7 is quite specific to the case of a system being $(n-1)$-dominant with a cycle-stable automaton (implying that the dominated component of the linearized dynamics is stable; see also Theorem 2.15 in Subsection 2.2.2). A similar result can be obtained for p-dominant systems using the celebrated formula of Ledrappier (1981), stating that the Haussdorff dimension of any compact invariant set of a dynamical system $\mathsf{Sys} = (\mathbb{R}^n, f)$ is bounded by m where $m \in \mathbb{N}$ is any integer such that the product of the m largest singular values of $\frac{\partial \chi}{\partial x}(T, x)$ is smaller than 1 for all $x \in \mathbb{R}^n$ and for some $T \in \mathbb{N}_{>0}$. The theory of dominance, by studying dominated splittings of the dynamics, provides algorithmic tools to answer this kind of questions; for the sake of brievety, the details are omitted here (see also the proof of Proposition 2.28 in Appendix A.2.10).

This concludes the example illustrating the concept of p-dominance for nonlinear dynamical systems. In the next subsection, we discuss the algorithmic aspects of the verification of p-dominance. Then, in Subsection 2.3.3, we present two applications of the theory of p-dominance for dynamical systems, namely regarding the verification of the property of hyperbolicity (see also Subsection 1.2.3) and the estimation of the topological entropy (see also Subsection 1.5.2) of these systems.

2.3.2 Algorithmic verification of p-dominance of smooth dynamical systems

In this subsection, we describe an algorithmic framework for the verification of p-dominance of dynamical systems. This framework draws on the algorithm for the verification of p-dominance for switched linear systems, described in Subsection 2.2.3, and on the technique of abstraction of dynamical systems, introduced in Section 1.4. The subsection is organized as follows. First, we present an algorithm to compute, if it exists, a set of symmetric matrices satisfying the second condition in the definition of p-dominance for dynamical systems (Definition 2.21) when the automaton Aut, the set of sets of matrices $\{\mathcal{A}_i\}_{i\in\Sigma}$ and the set of rates $\{\gamma_\theta\}_{\theta\in\Theta}$ are given. This algorithm extends the one presented in Subsection 2.2.3 for the verification of p-dominance of switched linear systems by considering automata whose edges are associated to sets of matrices instead of single matrices. Then, we explain how to build an automaton Aut and a set of sets of matrices $\{\mathcal{A}_i\}_{i\in\Sigma}$ satisfying the first condition in the definition of p-dominance for dynamical systems. For this step, we use the approach of abstraction of dynamical systems introduced in Section 1.4. Finally, we discuss the complexity of the overall algorithmic framework.

Computation of the symmetric matrices

Consider an automaton $\mathsf{Aut} = (Q, \Sigma, \Theta)$, a set of sets of matrices $\{\mathcal{A}_i\}_{i\in\Sigma} \Subset \mathbb{R}^{n\times n}$ and a set of rates $\{\gamma_\theta\}_{\theta\in\Theta} \subseteq \mathbb{R}_{>0}$. Computing a set of matrices $\{P_q\}_{q\in Q} \subseteq \mathbb{S}_p^{n\times n}$ satisfying Condition 2 in Definition 2.21 can be addressed by solving the following optimization problem: with variables $\{P_q\}_{q\in Q} \subseteq \mathbb{S}^{n\times n}$ and $\epsilon \in \mathbb{R}$,

$$\begin{aligned}
&\max && \epsilon && && \text{(2.9a)}\\
&\text{s.t.} && A^\top P_{\mathsf{t}(\theta)} A - \gamma_\theta^2 P_{\mathsf{s}(\theta)} \preceq -\epsilon I && \forall\, \theta \in \Theta,\ A \in \mathcal{A}_{\mathsf{i}(\theta)}, && \text{(2.9b)}\\
& && -I \preceq P_q \preceq I && \forall\, q \in Q, && \text{(2.9c)}\\
& && P_q \in \mathbb{S}_p^{n\times n} && \forall\, q \in Q. && \text{(2.9d)}
\end{aligned}$$

The constraint (2.9b) contains an infinite number of semidefinite constraints; we explain below how the problem can be modified to have a finite number of constraints. But, before that, let us mention that as for the case of switched linear systems, the constraints (2.9d) on the inertia of the matrices $\{P_q\}_{q\in Q}$ can be dropped without any impact on the outcome of the decision problem "is there a set of matrices $\{P_q\}_{q\in Q} \subseteq \mathbb{S}_p^{n\times n}$ satisfying the second condition in Definition 2.21?" This statement is formalized in the theorem below.

Theorem 2.23. *Consider an automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$*, a set of sets of matrices* $\{\mathcal{A}_i\}_{i\in\Sigma} \Subset \mathbb{R}^{n\times n}$ *and a set of rates* $\{\gamma_\theta\}_{\theta\in\Theta} \subseteq \mathbb{R}_{>0}$*. Let Assumption 2.16 hold (*Aut *is essential). Then, any optimal solution* $(\{P_q^\star\}, \epsilon^\star)$ *of* (2.9a)–(2.9c) *satisfies that* $\epsilon^\star > 0$ *and* $\{P_q^\star\}_{q\in Q} \subseteq \mathbb{S}_p^{n\times n}$ if and only if *there is a set of matrices* $\{P_q\}_{q\in Q} \subseteq \mathbb{S}_p^{n\times n}$ *satisfying Condition 2 in Definition 2.21 with* Aut, $\{\mathcal{A}_i\}_{i\in\Sigma}$ *and* $\{\gamma_\theta\}_{\theta\in\Theta}$.

Proof. The proof is identical to the proof of Corollary 2.18. □

Based on the above, we now restrict our attention to the subproblem (2.9a)–(2.9c). This problem is a semidefinite optimization problem, but (2.9b) contains an *infinite* number of semidefinite constraints. However, we will see that the problem can be tightened (i.e., modified with stronger constraints) to obtain a semidefinite optimization problem with a finite number of constraints. To do this, we assume that for each $i \in \Sigma$, the matrix set $\mathcal{A}_i$ can be enclosed in a convex set of the form

$$\mathcal{A}_i \subseteq A_i^{\mathrm{c}} + \operatorname{conv}\{A_{i,1}^{\mathrm{h}}, \ldots, A_{i,N_i}^{\mathrm{h}}\} + r_i\mathbb{B}, \quad \text{for each } i \in \Sigma, \tag{2.10}$$

where $A_i^{\mathrm{c}}, A_{i,1}^{\mathrm{h}}, \ldots, A_{i,N_i}^{\mathrm{h}} \in \mathbb{R}^{n\times n}$ and $r_i \geq 0$ for all $i \in \Sigma$, and $\mathbb{B} = \{A \in \mathbb{R}^{n\times n} : \|A\| \leq 1\}$.

Using these convex enclosures (2.10), we define the following semidefinite optimization problem: with variables $\{P_q\}_{q\in Q} \subseteq \mathbb{S}^{n\times n}$, $\{E_q\}_{q\in Q} \subseteq \mathbb{S}^{n\times n}$,

$\{\delta_q\}_{q\in Q} \subseteq [0,1]$, $\eta \in \mathbb{R}_{\geq 0}$ and $\epsilon \in \mathbb{R}$,

$$\max \quad \epsilon \tag{2.11a}$$

$$\text{s.t.} \quad {A^{\mathrm{c}}_{\mathsf{i}(\theta)}}^\top P_{\mathsf{t}(\theta)} A^{\mathrm{c}}_{\mathsf{i}(\theta)} + {A^{\mathrm{c}}_{\mathsf{i}(\theta)}}^\top P_{\mathsf{t}(\theta)} A^{\mathrm{h}}_{\mathsf{i}(\theta),j} + {A^{\mathrm{h}}_{\mathsf{i}(\theta),j}}^\top P_{\mathsf{t}(\theta)} A^{\mathrm{c}}_{\mathsf{i}(\theta)} + {A^{\mathrm{h}}_{\mathsf{i}(\theta),j}}^\top E_{\mathsf{t}(\theta)} A^{\mathrm{h}}_{\mathsf{i}(\theta),j} - \gamma_\theta^2 P_{\mathsf{s}(\theta)} + \eta I \preceq -\epsilon I \quad \forall\, \theta \in \Theta,\, j \in \{1,\ldots,N_{\mathsf{i}(\theta)}\}, \tag{2.11b}$$

$$\eta \geq r_{\mathsf{i}(\theta)}^2 \delta_{\mathsf{t}(\theta)} + 2 r_{\mathsf{i}(\theta)} \| P_{\mathsf{t}(\theta)} (A^{\mathrm{c}}_{\mathsf{i}(\theta)} + A^{\mathrm{h}}_{\mathsf{i}(\theta),j}) \| \quad \forall\, \theta \in \Theta,\, j \in \{1,\ldots,N_{\mathsf{i}(\theta)}\}, \tag{2.11c}$$

$$-I \preceq P_q \preceq \delta_q I,\; P_q \preceq E_q,\; E_q \succeq 0 \quad \forall\, q \in Q. \tag{2.11d}$$

Note that the constraints (2.11c) can be expressed as semidefinite constraints, using the Schur complement: for any $z \in \mathbb{R}$ and $B \in \mathbb{R}^{n\times n}$, it holds that $z \geq \|B\|$ if and only if $\begin{bmatrix} zI & B \\ B^\top & zI \end{bmatrix} \succeq 0$ (see, e.g., Ben-Tal and Nemirovski, 2001, p. 152). The following theorem states that any feasible solution of (2.11) provides a feasible solution of (2.9a)–(2.9c). (See also the paragraph below this theorem for an explanation of the derivation of this tightened optimization problem.)

Theorem 2.24. *Consider an automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$, *a set of sets of matrices* $\{\mathcal{A}_i\}_{i\in\Sigma} \in \mathbb{R}^{n\times n}$ *and a set of rates* $\{\gamma_\theta\}_{\theta\in\Theta} \subseteq \mathbb{R}_{>0}$. *Let* $\{A^{\mathrm{c}}_i\}_{i\in\Sigma}$, $\{A^{\mathrm{h}}_{i,j}\}_{i\in\Sigma,\, j\in\{1,\ldots,N_i\}}$ *and* $\{r_i\}_{i\in\Sigma}$ *be as in* (2.10). *Let* $(\{P_q\}_{q\in Q}, \{E_q\}_{q\in Q}, \{\delta_q\}_{q\in Q}, \eta, \epsilon)$ *be a feasible solution of* (2.11b)–(2.11d). *Then, it holds that* $(\{P_q\}_{q\in Q}, \epsilon)$ *is a feasible solution of* (2.9b)–(2.9c).

Proof. See Appendix A.2.8. □

The idea behind the derivation of (2.11) as a tightening of (2.9a)–(2.9c) is the following. For any $\theta \in \Theta$, by replacing the matrix $P_{\mathsf{t}(\theta)}$ by the matrix $E_{\mathsf{t}(\theta)} \succeq 0$, the constraint

$${A^{\mathrm{c}}_{\mathsf{i}(\theta)}}^\top P_{\mathsf{t}(\theta)} A^{\mathrm{c}}_{\mathsf{i}(\theta)} + {A^{\mathrm{c}}_{\mathsf{i}(\theta)}}^\top P_{\mathsf{t}(\theta)} B + B^\top P_{\mathsf{t}(\theta)} A^{\mathrm{c}}_{\mathsf{i}(\theta)} + B^\top E_{\mathsf{t}(\theta)} B - \gamma_\theta^2 P_{\mathsf{s}(\theta)} \preceq -\epsilon I \tag{2.11b'}$$

provides a convex enclosure of the constraint (2.9b) with this θ, in the sense that, for fixed $P_{\mathsf{s}(\theta)}$, $P_{\mathsf{t}(\theta)}$, $E_{\mathsf{t}(\theta)}$ and ϵ, the set of matrices B that satisfy the constraint (2.11b'), shifted by $A^{\mathrm{c}}_{\mathsf{i}(\theta)}$, is a convex set containing the set of matrices A that satisfy (2.9b) with this θ. Hence, it suffices to impose that this set contains the vertices of the convex enclosure (2.10) of $\mathcal{A}_{\mathsf{i}(\theta)}$. This is achieved by imposing that the matrix inequality (2.11b') is satisfied at $B = A^{\mathrm{h}}_{\mathsf{i}(\theta),j}$ for all $j \in \{1,\ldots,N_{\mathsf{i}(\theta)}\}$ with a margin taking into account the deviation of norm $r_{\mathsf{i}(\theta)}$ around $A^{\mathrm{h}}_{\mathsf{i}(\theta),j}$ (see (2.10)). This margin is captured by the term ηI where η satisfies (2.11c). See also the proof of Theorem 2.24 (Appendix A.2.8) for details.

Summarizing, Theorems 2.23 and 2.24 show that if the automaton, the set of sets of matrices and the set of rates are given, then the verification of p-dominance for a given dynamical system can be reduced to a semidefinite optimization problem, and thus can be solved efficiently using classical convex optimization algorithms. In the next subsubsection, we explain how to compute an automaton and a set of sets of matrices satisfying the first condition in the definition of p-dominance for dynamical systems. We also discuss the choice of the set of rates.

Computation of the automaton, the sets of matrices and the rates

The computation of the automaton can be achieved by using the technique of abstraction introduced in Section 1.4 and which amounts to compute a finite representation, called a *symbolic model*, of the dynamical system. For a reminder, given a dynamical system $\mathsf{Sys} = (\mathbb{R}^n, f)$ and a subset $\Lambda \subseteq \mathbb{R}^n$, a *symbolic model* of Sys on Λ (with time step $T = 1$) consists of a finite set of regions $\{O_q\}_{q\in Q} \Subset \mathbb{R}^n$ satisfying $\Lambda \subseteq \bigcup_{q\in Q} O_q$, and a set of edges $E \subseteq Q \times Q$ representing the possible transitions of the system among the different regions, meaning that $(q_1, q_2) \in E$ if $\{f(x) : x \in O_{q_1}\} \cap O_{q_2} \neq \emptyset$ (see Definition 1.57 in Subsection 1.4.1). All symbolic models considered in this section are assumed to have time step equal to one, and thus we refer to them simply as *symbolic models*. See Figure 2.12 for an illustration of a symbolic model built with the method presented in Subsection 1.4.2.

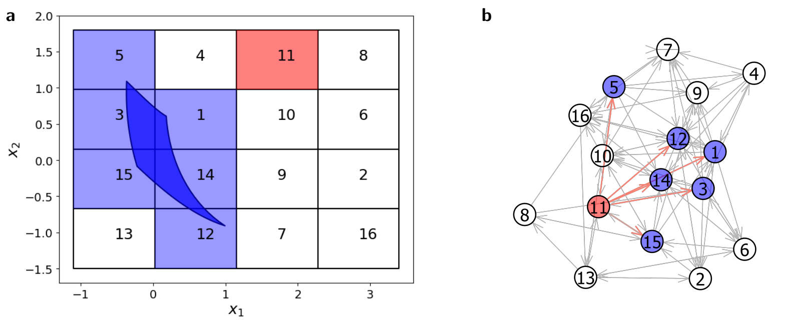


Figure 2.12: Symbolic model for the *Ikeda* dynamical system (see Example 2.8 in Subsection 2.3.3) on the set $\Lambda = [-1.1, 3.4] \times [-1.5, 1.8]$. **a**: Discretization of Λ into 16 rectangular regions O_q, indexed by $Q = \{1, \ldots, 16\}$. The image of the region O_{11} (in red) is represented in dark blue. The different regions that intersect the image of O_{11} are represented in light blue. **b**: Graph representing the transitions (edges) between the different regions of the symbolic model. The outgoing edges from the node 11 are highlighted in red.

Given a symbolic model $(\{O_q\}_{q\in Q}, E)$ of $\mathsf{Sys} = (\mathbb{R}^n, f)$ on $\Lambda \subseteq \mathbb{R}^n$, we first build an automaton as follows. We let Aut be the automaton defined by (Q, Σ, Θ) where $\Sigma = Q$ and $\Theta = \{(q_1, q_1, q_2) : (q_1, q_2) \in E\} \subseteq Q \times \Sigma \times Q$ (in other words, the transitions of Aut are given by the edges of E labeled with the source node of the edge in question). Then, we define the sets of matrices $\{\mathcal{A}_i\}_{i\in\Sigma}$ as follows. For each $q \in Q = \Sigma$, we let $\mathcal{A}_q \subseteq \mathbb{R}^{n\times n}$ be a set of matrices such that for every $x \in O_q$, $\frac{\partial f}{\partial x}(x) \in \mathcal{A}_q$. This set of matrices can be defined for instance by using the Lipschitz continuity of $\frac{\partial f}{\partial x}$ (which was assumed in the construction of the symbolic model; see Assumption 1.63 in Subsection 1.4.2) or by identifying the variables that influence the value of $\frac{\partial f}{\partial x}$ and how these variables vary in the region in question (as in the example at the end of Subsection 2.3.1). By construction, Aut and $\{\mathcal{A}_i\}_{i\in\Sigma}$ defined as above satisfy the first condition in the definition of p-dominance of Sys.

Proposition 2.25. *Consider a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$ *and a set* $\Lambda \subseteq \mathbb{R}^n$. *Let* $(\{O_q\}_{q\in Q}, E)$ *be a symbolic model of* Sys *on* Λ, *and let* $\mathsf{Aut} = (Q, \Sigma, \Theta)$ *and* $\{\mathcal{A}_i\}_{i\in\Sigma} \Subset \mathbb{R}^{n\times n}$ *be defined as above. Then,* Aut *and* $\{\mathcal{A}_i\}_{i\in\Sigma}$ *satisfy Condition 1 in Definition 2.21.*

Proof. Straightforward from the definition of a symbolic model and from the construction of Aut and $\{\mathcal{A}_i\}_{i\in\Sigma}$. □

Remark 2.8. In practice, the definition of the sets of matrices $\{\mathcal{A}_i\}_{i\in\Sigma}$ often directly provides convex enclosures in the form of (2.10); but this is not required and can be achieved in a second time if one wants to use Theorem 2.24 to verify that the second condition in Definition 2.21 holds.

Now that we have seen how to construct an automaton and a set of sets of matrices satisfying the first condition in the definition of p-dominance, it remains to discuss how to find, if it exists, a set of rates satisfying the second condition in the definition of p-dominance with a given automaton and a given set of sets of matrices. As in the case of switched linear systems, there is in general no systematic way to find such a set of rates, but we can nevertheless reduce the search space for the rates by using constraints similar to the ones in Proposition 2.19.

Proposition 2.26. *Consider an automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$, *a set of sets of matrices* $\{\mathcal{A}_i\}_{i\in\Sigma} \Subset \mathbb{R}^{n\times n}$, *a set of rates* $\{\gamma_\theta\}_{\theta\in\Theta} \subseteq \mathbb{R}_{>0}$ *and a set of matrices* $\{P_q\}_{q\in Q} \subseteq \mathbb{S}^{n\times n}_p$ *satisfying the second condition in Definition 2.21. Let* $(\theta_t)_{t=0}^{T-1}$ *be a cycle in* Aut, *and for each* $t \in \{0, \ldots, T-1\}$, *let* $A_t \in \mathcal{A}_{\mathsf{i}(\theta_t)}$. *Define* $\bar{A} = A_{T-1} \cdots A_0$ *and* $\bar{\gamma} = \gamma_{\theta_0} \cdots \gamma_{\theta_{T-1}}$. *Then, it holds that* $\bar{A}$ *has* p *eigenvalues with modulus* $|\lambda_i| > \bar{\gamma}$ *and* $n - p$ *eigenvalues with modulus* $|\lambda_i| < \bar{\gamma}$.

Proof. The proof is identical to the proof of Condition 1 in Proposition 2.19. □

In all generality, the rates can be different for each transition of the automaton. However, when the size of the automaton increases, it becomes rapidly intractable to explore the space of all sets of rates having a different value for each transition of the automaton. Therefore, in the numerical experiments, we often search a suitable set of rates with the additional requirement that all rates are the same and we use Proposition 2.26 to derive bounds on the common value of the rates (see also the example at the end of Subsection 2.3.1 for an illustration); the price to pay is to increase the conservatism of the algorithmic framework for the verification of p-dominance of dynamical systems. However, for some applications, assuming that all rates are equal, or even fixing the value of the rates, does not add any conservatism for the numerical analysis of the application in question. This is the case for instance for the verification of hyperbolicity of dynamical systems, for which we may assume that all rates are equal to one, as we will see in Subsection 2.3.3.

Complexity of the overall algorithmic framework

The complexity of the overall algorithmic framework is mainly driven by the complexity of computing a symbolic model of the system. For a given diameter of the regions, this grows in the worst case as a power of the dimension of the system (this is the *curse of dimensionality* of the abstraction approach; see, e.g., Reißig, 2011, p. 2588). Once a symbolic model is computed, one can construct an automaton and a set of sets of matrices—or more precisely convex enclosures of these sets of matrices—with the approach described in the previous subsubsection. If for instance the Lipschitz continuity assumption is used to construct the sets of matrices, this step can be performed in linear time with respect to the number of regions in the symbolic model.

Regarding the selection of the rates, there is no clear complexity for this step as we do not have a systematic methodology for it (see the previous subsubsection). If we assume that all rates are equal and assume some robustness of the property of p-dominance of the dynamical system, then the rates can be selected by trying equally spaced values within the confidence interval defined by the constraints in Proposition 2.26.

Finally, for a given automaton $\mathsf{Aut} = (Q, \Sigma, \Theta)$, a given set of convex sets of matrices in the form of (2.10) and a given set of rates, the optimization problem (2.11) can be solved efficiently using any polynomial-time semidefinite optimization algorithm. The optimization problem involves $2|Q|$ matrix variables of dimension $n \times n$ and roughly $|Q|$ scalar variables, and $4|Q| + 2|\Theta|\bar{N}$

semidefinite constraints of size $n \times n$ or $2n \times 2n$, where $\bar{N} = \max_{i \in \Sigma} N_i$.

2.3.3 Numerical examples and applications

In this subsection, we present two applications of the theory of p-dominance for dynamical systems. The first application deals with the verification of the property of hyperbolicity of dynamical systems and the second one with the estimation of the topological entropy of dynamical systems. Both applications are illustrated with numerical examples.

p-dominance and hyperbolicity of dynamical systems

In Subsection 1.2.3, we introduced the concept of hyperbolicity for invertible dynamical systems and discussed its applications regarding the structural stability of its invariant sets. For a reminder, an invariant set of an invertible dynamical system is *hyperbolic* if for any point in the set there are two complementary subspaces, one being stable for the linearized dynamics in forward time and one being stable for the linearized dynamics in backward time (see Definition 1.31).

It turns out that the property of hyperbolicity can be verified algorithmically using the framework of p-dominance for dynamical systems. Moreover, in that case, the rates of p-dominance can be fixed to be equal to one, which reduces considerably the complexity of the overall algorithmic framework for the verification of p-dominance. The connection between hyperbolicity and p-dominance is explained in the following proposition.

Proposition 2.27. *Consider an invertible dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$ *and a set* $\Lambda \subseteq \mathbb{R}^n$ *invariant for* Sys. *Consider the following propositions:*

1. *There is* $p \in \{0, \ldots, n\}$ *such that* Sys *is* p*-dominant on* Λ *with some automaton* (Q, Σ, Θ) *and with set of rates* $\{\gamma_\theta\}_{\theta \in \Theta}$ *uniformly equal to one (i.e.,* $\gamma_\theta = 1$ *for all* $\theta \in \Theta$*).*

2. Λ *is a hyperbolic invariant set for* Sys.

It holds that $1 \Rightarrow 2$. *Moreover, if* Λ *is bounded and connected, then* $1 \Leftrightarrow 2$.

Proof. See Appendix A.2.9. □

The numerical example below illustrates the use of the theory of p-dominance for the identification of hyperbolic invariant sets of invertible dynamical systems.

Example 2.8. Consider the dynamical system $\mathsf{Sys} = (\mathbb{R}^2, f)$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the *modified Ikeda map*, defined by

$$f\left(\begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}\right) = \begin{bmatrix} r + a(x^{(1)}\cos\tau - x^{(2)}\sin\tau) \\ b(x^{(1)}\sin\tau + x^{(2)}\cos\tau) \end{bmatrix}, \qquad \tau = \frac{C_1 - C_3}{1 + (x^{(1)})^2 + (x^{(2)})^2},$$

where $r = 2$, $C_1 = 0.4$, $C_3 = 6$, $a = 0.9$, $b = -0.9$. The modified Ikeda map is known to be invertible and to have a nonempty hyperbolic invariant set inside $\Omega \doteq [-1.1, 3.4] \times [-1.5, 1.8]$; see, e.g., Osipenko (2007, Example 152).

We use the framework described in Subsection 2.3.2 to show that the maximal invariant set Λ of Sys included in Ω is hyperbolic. To do that, first, we compute symbolic models of Sys on Ω using the method described in Subsection 1.4.2. From these symbolic models and by using Proposition 1.67 in Subsection 1.4.3, we obtain over-approximations of Λ; see Figure 2.13-a (the finer the abstraction the more accurate is the over-approximation of the maximal invariant set).

Then, we show that Λ is a hyperbolic invariant set for Sys. Therefore, we use Proposition 2.27 and show that Sys is 1-dominant, with rates uniformly equal to one, on a subset Λ' containing Λ. More precisely, we use the over-approximation of Λ provided by the abstraction with $h = [0.015, 0.014]^\top$ in Figure 2.13-a (which has 1744 nodes and 7048 edges). From this abstraction, we build an automaton $\mathsf{Aut} = (Q, \Sigma, \Theta)$ and a set of sets of matrices $\{\mathcal{A}_i\}_{i\in\Sigma}$ satisfying the first condition in Definition 2.21 with the approach presented in Subsection 2.3.2. Then, we solve the optimization problem (2.11) with Aut and $\{\mathcal{A}_i\}_{i\in\Sigma}$, and with rates $\{\gamma_\theta\}_{\theta\in\Theta}$ uniformly equal to one. This provides a set of symmetric matrices $\{P_q\}_{q\in Q} \subseteq \mathbb{S}_1^{2\times 2}$ satisfying the second condition in Definition 2.21, thereby showing that Sys is 1-dominant on Λ, with rates uniformly equal to one. For the interested reader, the quadratic 1-cones associated to the matrices $\{P_q\}_{q\in Q}$, along a trajectory of the system, are represented in Figure 2.13-b. We observe that the cones are contracted by the prolonged system $\partial\mathsf{Sys}$, as predicted by Proposition 2.9 in Subsection 2.2.2.

Related works. The problem of algorithmic verification of hyperbolicity of dynamical systems was addressed by Osipenko (2007); this is the only other work on this problem we are aware of. Osipenko's approach relies on constructing an *abstraction of the prolonged system.* The Morse spectrum of the system can then be over-approximated by bounding the minimal and maximal growth rate of the derivative along cycles in the graph of the abstraction. A certificate of hyperbolicity of the system is then obtained if the over-approximation of the Morse spectrum keeps away from zero. Regarding the computational complexity, this approach also suffers from the curse of dimensionality since it

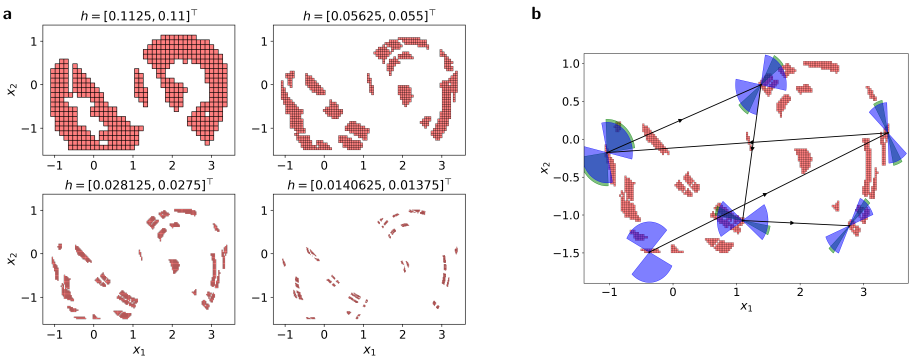


Figure 2.13: Computation of a hyperbolic invariant set for the Ikeda map inside $\Omega = [-1.1, 3.4] \times [-1.5, 1.8]$ (see Example 2.8). **a**: Four abstractions of the system on Ω, constructed with the method presented in Subsection 1.4.2. The step size of the rectangular discretizations are indicated above the plots. The regions in red provide over-approximations Λ' of the maximal invariant set Λ of the system inside Ω. The finer the discretization, the closer is the over-approximation to Λ. **b**: Quadratic 1-cones (in blue) computed with the algorithm presented in Subsection 2.3.2 for the verification of 1-dominance of the system on Λ'. The cones have been represented along a trajectory of the system (in black). The images of the cones by the derivative of the system are represented in green. We observe that the images are contained in the cone at the next point of the trajectory, as predicted by Proposition 2.9.

requires to compute an abstraction of a system with dimension $2n-1$ (where n is the dimension of the original system). It is difficult to have a further comparison between the two methods because their respective complexity will depend on the size of the abstraction of the system on its maximal invariant set Λ, which can be smaller than $\mathcal{O}(\|h\|^{-n})$ if Λ is low-dimensional, where h is the discretization step of the abstraction.

p-dominance and topological entropy of dynamical systems

In Subsection 1.5.2, we introduced the concept of topological entropy for hybrid systems, and in Subsection 2.2.4, we showed that the theory of p-dominance for discrete-time switched linear systems can be useful to obtain bounds on the topological entropy of these systems. In this subsubsection, we show that the latter approach can be generalized to dynamical systems.

Proposition 2.28. *Consider a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$ *and let* $\Lambda \subseteq \mathbb{R}^n$ *be a bounded forward invariant set for* Sys*. Let* $h_{\mathrm{top}}(\mathsf{Sys}, \Lambda)$ *be the topological entropy of* Sys *restricted to* Λ *with cost function* $(x_1, x_2) \mapsto \|x_1 - x_2\|$*. Let* $p_1, \ldots, p_n \in \mathbb{N}$ *be such that* $p_k \leq k-1$ *for all* $k \in \{1, \ldots, n\}$*, and assume that* *for each* $k \in \{1, \ldots, n\}$, Sys *is* p_k*-dominant on* Λ *with automaton* $\mathsf{Aut}^{(k)} = (Q^{(k)}, \Sigma^{(k)}, \Theta^{(k)})$ *and set of rates* $\{\gamma_\theta^{(k)}\}_{\theta \in \Theta^{(k)}} \subseteq \mathbb{R}_{>0}$. *Then, it holds that*

$$h_{\mathrm{top}}(\mathsf{Sys}, \Lambda) \leq \sum_{k=1}^{n} \max\{0, \log_2 \hat{\gamma}_{\max}^{(k)}\},$$

where $\hat{\gamma}_{\max}^{(k)} = \max\{\, (\gamma_{\theta_0}^{(k)} \cdots \gamma_{\theta_{T-1}}^{(k)})^{1/T} : (\theta_t)_{t=0}^{T-1}$ *is a cycle in* $\mathsf{Aut}^{(k)}\,\}$. *Similarly, if*[11] $p_1, \ldots, p_m \in \mathbb{N}$ *are such that* $p_k \geq k$ *for all* $k \in \{1, \ldots, m\}$, *and for each* $k \in \{1, \ldots, m\}$, Sys *is* p_k*-dominant on* Λ *with automaton* $\mathsf{Aut}^{(k)} = (Q^{(k)}, \Sigma^{(k)}, \Theta^{(k)})$ *and set of rates* $\{\gamma_\theta^{(k)}\}_{\theta \in \Theta^{(k)}} \subseteq \mathbb{R}_{>0}$. *Then, it holds that*

$$h_{\mathrm{top}}(\mathsf{Sys}, \Lambda) \geq \sum_{k=1}^{m} \max\{0, \log_2 \hat{\gamma}_{\min}^{(k)}\},$$

where $\hat{\gamma}_{\min}^{(k)} = \min\{\, (\gamma_{\theta_0}^{(k)} \cdots \gamma_{\theta_{T-1}}^{(k)})^{1/T} : (\theta_t)_{t=0}^{T-1}$ *is a cycle in* $\mathsf{Aut}^{(k)}\,\}$.

Proof. See Appendix A.2.10. □

For practical use of the upper bound in Proposition 2.28, it should be noted that a dynamical system is always 0-dominant on any bounded forward invariant set, with respect to some sufficiently large set of rates. For instance, if $\Lambda \subseteq \mathbb{R}^n$ is a bounded forward invariant set for a dynamical system $\mathsf{Sys} = (\mathbb{R}^n, f)$, then we can take the set of rates being larger than $\sup_{x \in \Lambda} \|\frac{\partial f}{\partial x}(x)\|$. In this case, the upper bound in Proposition 2.28, with $p_k = 0$ for each $k \in \{1, \ldots, n\}$, coincides with the classical upper bound $h_{\mathrm{top}}(\mathsf{Sys}, \Lambda) \leq n \sup_{x \in \Lambda} \log_2 \|\frac{\partial f}{\partial x}(x)\|$ (see, e.g., Liberzon and Mitra, 2018, Proposition 2). The upper bound in Proposition 2.28 refines this bound when Sys is p-dominant with $p < n$. Similarly, the lower bound in Proposition 2.28 refines the classical lower bound $h_{\mathrm{top}}(\mathsf{Sys}, \Lambda) \geq \sup_{x \in \Lambda} \log_2 |\det \frac{\partial f}{\partial x}(x)|$ (see, e.g., Liberzon and Mitra, 2018, Proposition 3) when Sys is p-dominant with $p < n$.

Also, let us mention that the bounds on the topological entropy in Proposition 2.28 are tight in the case of LTI systems. Indeed, on the one hand, the topological entropy of a LTI system $(\mathbb{R}^n, A)$, with $A \in \mathbb{R}^{n \times n}$, is equal to $\sum_{|\lambda_i| > 1} \log_2 |\lambda_i|$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A (see, e.g., Matveev and Savkin, 2009, Theorem 2.4.2). On the other hand, from the characterization of p-dominant LTI systems (see Proposition 2.3 in Subsection 2.2.1), the values of the rate of dominance, for $k = 1, \ldots, n$, can be taken arbitrarily close to the modulus of the eigenvalues λ_k of A, so that the bounds in Proposition 2.28 provide tight estimates of the topological entropy of the system.

The numerical example below illustrates the use of the theory of p-dominance for the estimation of the topological entropy of dynamical systems.

[11] Note that m may be smaller than n here.

Example 2.9. Consider the dynamical system $\mathsf{Sys} = (\mathbb{R}^2, f)$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f\left(\begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}\right) = \frac{1}{((x^{(1)})^2 + (x^{(2)})^2)^{3/4}} \begin{bmatrix} (x^{(1)})^2 - (x^{(2)})^2 \\ 2x^{(1)}x^{(2)} \end{bmatrix}.$$

It is easily seen that any set of the form $\{x \in \mathbb{R}^n : r_1 \leq \|x\| \leq r_2\}$, with $0 < r_1 < 1 < r_2$, is forward invariant for Sys. Hence, for definiteness, let $\Lambda = \{x \in \mathbb{R}^n : \frac{1}{2} \leq \|x\| \leq 2\}$. Using the theory of p-dominance, we will show that $0.88 \leq h_{\mathrm{top}}(\mathsf{Sys}, \Lambda) \leq 1.11$.

Indeed, it can be shown that Sys is 1-dominant on Λ with set of rates uniformly equal to 1 and also with set of rates uniformly equal to 1.85. Also, Sys can be shown to be 0-dominant on Λ with set of rates uniformly equal to 2.15. These results were obtained by using the method described in Subsection 2.3.2 (the resulting automaton has 724 states and 4080 transitions). Hence, by Proposition 2.28, we get the claimed bounds on the topological entropy of Sys restricted to Λ:

$$0.88 \leq \log_2 1.85 \leq h_{\mathrm{top}}(\mathsf{Sys}, \Lambda) \leq \log_2 1 + \log_2 2.15 \leq 1.11,$$

concluding the example.

Related works. The approach of Proposition 2.28 for the estimation of the topological entropy of dynamical systems connects with the upper bound in Matveev and Pogromsky (2019, Theorem 11) for the topological entropy of a dynamical systems $\mathsf{Sys} = (\mathbb{R}^n, f)$ on a bounded forward invariant subset $\Lambda \subseteq \mathbb{R}^n$:

$$h_{\mathrm{top}}(\mathsf{Sys}, \Lambda) \leq \limsup_{T\to\infty} \frac{1}{T} \left(\sup_{x\in\Lambda} \sum\nolimits_{i=1}^{n} \max\left\{0, \log_2 \rho_i\left(\tfrac{\partial\chi}{\partial x}(T, x)\right)\right\}\right), \quad (2.12)$$

where $\rho_1(A), \ldots, \rho_n(A)$ denote the singular values of $A \in \mathbb{R}^{n\times n}$. In fact, the proof of the upper bound in Proposition 2.28 (see Appendix A.2.10) relies on the observation that, if Sys is p-dominant with $\mathsf{Aut} = (Q, \Sigma, \Theta)$ and $\{\gamma_\theta\}_{\theta\in\Theta}$, then for T large enough, an upper bound on the $n - p$ smallest singular values of $\frac{\partial\chi}{\partial x}(T, x)$ can be derived from the set of rates: namely, this upper bound is given by $\hat{\gamma}_{\max,T} = \max\{\, \gamma_{\theta_0} \cdots \gamma_{\theta_{T-1}} : (\theta_t)_{t=0}^{T-1} \text{ is a cycle in } \mathsf{Aut} \,\}$. The upper bound in Proposition 2.28 is less sharp than (2.12) because $\hat{\gamma}_{\max,T}$ only provides an upper bound on the singular values of $\frac{\partial\chi}{\partial x}(T, x)$, but it has the advantage of providing a way of computing such upper bounds via the framework of p-dominance.

2.4 Conclusions

In this chapter, we studied the property of having a dominated p-splitting for discrete-time switched linear systems and discrete-time smooth dynamical systems, that is, the property that their linearized dynamics can be decomposed into a p-dimensional dominant dynamics and a complementary $(n - p)$-dimensional dominated dynamics (with n the dimension of the system). The asset of this separation feature is that it allows to refine the analysis of the dynamics on each component. For instance, in quantized control, it allows to use a different level of quantization for each component, thereby providing refined bounds on the topological entropy of the system compared to the ones that are generally obtained by analyzing the dynamics as a single "isotropic" component. Another example is for the study of the convergence of the normalized trajectories of switched linear systems. From the normalization, it holds that only the dominant component of the dynamics will be relevant for the asymptotic behavior of the system, so that the normalized trajectories converge toward a subspace of dimension at most p, independently of their initial condition. In the case of 1-dominance, this translates as the incremental stability of the normalized trajectories, and thus allows to study the properties of incremental stability using a switched systems approach instead of the classical differential Lyapunov approaches based on the normalized system (which would require to find a Lyapunov function for the nonlinear prolonged system). Finally, another application discussed in this chapter is for the verification of the property of hyperbolicity, a ubiquitous concept in systems theory allowing for instance to study the structural stability of attractors of dynamical systems or the quantized control of these systems.

We also provided an algorithmic framework for the verification of the property of having a dominated p-splitting, drawing on a geometric characterization of this property. This characterization is formulated in terms of the contraction properties, by the linearized dynamics, of a set of quadratic p-cones; the contraction relations of the cones being driven by the transitions of an automaton representing the admissible "traces" of the system (either the switching signals in the case of switched linear systems, or the trajectories of the system in the case of dynamical systems). The use of quadratic p-cones allows to encode the satisfiability of the geometric condition as the feasibility of a set of matrix inequalities, which results in a tractable criterion for the verification of the property of domination (thanks to the well-established theory of conic optimization).

The concept of p-dominance was initially introduced by Forni and Sepulchre (2019) for the study of continuous-time smooth dynamical systems whose linearized dynamics can be decomposed into p "slow" modes and $n - p$ "fast" modes (with n the dimension of the system). Our work draws on this seminal work, but differs in that the geometric characterization in Forni and Sepulchre (2019) relies on a single quadratic p-cones that is contracted by the linearized dynamics while our characterization involves several quadratic p-cones. The conclusions in Forni and Sepulchre (2019) on the asymptotic behavior of systems satisfying the condition of p-dominance with a single quadratic p-cone are also stronger; in particular, it holds that the asymptotic dynamics of these systems is essentially the one of a p-dimensional system. The notion of p-dominance considered in this thesis does not allow to derive such a property (for instance there are examples of hyperbolic dynamical systems having a complex asymptotic dynamics); however, it allows to capture a larger class of systems.

An interesting direction for further research is to investigate potential applications of the property of a dominated splitting for the symbolic control of dynamical systems. Indeed, the technique of abstraction is known to suffer from the curse of dimensionality, which means that the complexity of constructing and analyzing symbolic models of dynamical systems grows exponentially with the dimension of the system. We believe that the theory of dominance can be useful to fight the curse of dimensionality when the linearized dynamics of the system has a low-dimensional dominant behavior. For instance, the property of hyperbolicity has already proved useful for the symbolic analysis of dynamical systems, namely via the notion of "Markov partition", which allows one to define a discretization of the state space that is adapted to the system (see, e.g., Robinson, 1999, Chapter 10). However, as we have seen, the question of algorithmic verification of the property of hyperbolicity has not received much attention so far. Hence, it could be worth investigating the possibility of combining the algorithmic framework for the verification of hyperbolicity with the above observations to systematize the use of the property of hyperbolicity in symbolic control algorithms, as well as in other computational problems in control. Another interesting direction for further research is to investigate possible extensions to other classes of hybrid systems, like switched linear systems with state-dependent switching and nonlinear hybrid systems. Notions related to dominated splittings, like incremental stability or positivity, are known to present subtleties when extended to hybrid systems (see, e.g., Postoyan et al., 2015, and Lanotte and Maggiolo-Schettini, 2005). However, the cases studied in this chapter provide proofs-of-concept that an algorithmic approach is possible to study these questions for some classes of hybrid systems, and it would be interesting to see how it can be extended to other classes of such systems.

Chapter 3

Quantized control of hybrid systems

In this chapter, we present the second part of our contributions, which deals with the observation and control of switched linear systems under communication constraints. As we will see, these questions connect with several other concepts in systems and control theory and in mathematics, such as topological entropy (introduced in Subsection 1.5.2 and touched on in the previous chapter), joint spectral radius (introduced in Subsection 1.3.2), and exterior algebra.

3.1 Introduction and literature review

This chapter addresses two important and challenging aspects of modern control systems, namely *quantization* and *switching*. Indeed, many modern control systems (such as IoT, networked systems, etc.) involve spatially distributed components that communicate through a shared, digital communication network, which can carry only a limited amount of information per unit of time. The limitation on the information flow can have large negative effects on the performance of the control loop. This has motivated a considerable amount of research to study control problems subject to data-rate constraints; see, e.g., Hespanha et al. (2007), Matveev and Savkin (2009) and Zhong-Ping and Teng-Fei (2013) for surveys. Furthermore, many systems and phenomenons encountered in modern applications involve switching among different operation modes (for instance cyber-physical systems, physical systems with impact, etc.). The presence of switching dynamics turns out to considerably complexify the analysis of these systems, even for simple models (see, e.g., Jungers, 2009). This has motivated the development of a new chapter in control theory, addressing switching phenomenons in control problems; see, e.g., van der Schaft and Schumacher (2000), Liberzon (2003) and Lin and Antsaklis (2009).

Although switched systems and control with data-rate constraints have been two active areas of research for some time now, the combination of these two aspects in control problems has started to receive attention in the literature only recently (some specific references are cited below). Combining these two aspects in a unified framework is however essential to tackle control problems encountered in a wide range of modern applications. For the main part, the works on networked switched systems can be split into two settings, depending on the assumption on the knowledge of the operating mode of the system by the different elements of the network: in the first setting, called *mode-dependent* quantized control, it is assumed that the current mode of the switched system is known by every element of the network, and in the second setting, called *mode-oblivious* quantized control, it is assumed that the mode of the system is not known by every elements of the network and is thus part of the information to be quantized and transmitted over the network.

In this thesis, we study control problems involving switching and data-rate constraints in the case of switched linear systems. In particular, we study the questions of state estimation and stabilization of switched linear systems under data-rate constraints, for the two settings discussed above. We detail our contributions and review the relevant literature in the paragraphs below.

Mode-dependent quantized control

First, we study the problem of quantized observation of linear time-varying (LTV) systems. In particular, we are interested in determining the minimal data rate required for state estimation of LTV systems, that is, the smallest data rate at which information needs to be sent by a coder to a decoder to estimate the state of the system with arbitrary finite accuracy; see also Figure 3.1 for an illustration. LTV systems can be seen as switched linear systems whose switching signal is fixed. We are thus in the mode-dependent setting since the mode of the system is known at all times by all elements of the network.

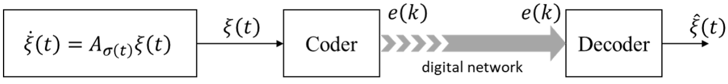


Figure 3.1: Mode-dependent quantized observation of LTV systems.

Inspired by the works of Shannon on information entropy and data rate requirements for reliable communication, it was soon realized that the question of the minimal data rate for state estimation of dynamical systems has strong connections with the notion of topological entropy. For a reminder, the topological entropy is a quantity that measures the growth rate of the smallest number of functions necessary to approximate the trajectories of a dynamical system with arbitrary finite accuracy (see Subsection 1.5.2). The study of the topological entropy and its link with the minimal data rate for state estimation has attracted a lot of attention from the control community in the last decades; see, e.g., Savkin (2006), Matveev and Pogromsky (2016) and Kawan (2018). A large part of these works focus on linear time-invariant systems or time-invariant dynamical systems with compact state space. For these systems, it is known that the topological entropy is equal to the minimal data rate for state estimation (see, e.g., Matveev and Pogromsky, 2016, or Proposition 1.80 in Subsection 1.5.2). These results are sometimes referred to as *data-rate theorems* (see, e.g., Kawan et al., 2021, p. 1). On the other hand, beyond these classes of systems, the situation is unfortunately much more elusive. The topological entropy is only known to be a lower bound on the minimal data rate for state estimation. In particular, it seems an open question whether the topological entropy is also an upper bound on to the minimal data rate for state estimation. As an evidence of this, we refer to Matveev and Pogromsky (2016) and Kawan (2018), where the lower bound is discussed but no proof, or counter-example, for the upper bound is presented.

Our main contribution regarding the quantized observation of LTV systems is to show that the equivalence of topological entropy and minimal data rate for state estimation extends to LTV systems. The relevance of this result is first theoretical, as it extends the data-rates theorems of linear time-invariant systems to LTV systems. Moreover, the proof of the theorem is constructive, as it provides a coder–decoder that observes the state of the system with arbitrary accuracy, and whose data rate can be as close as desired to the topological entropy of the system. The implementation of the coder–decoder with optimal data rate may however require unbounded memory, which can limit its practical usability. The result and its proof can nevertheless be useful for practical applications. For instance, it shows that the topological entropy can be used as a benchmark to evaluate the performance of different implementations of coders–decoders. Furthermore, the ideas presented in the proof of the theorem can be used to obtain efficient coders–decoders satisfying memory limitations, though possibly operating at suboptimal data rates.

We are also interested in the computation of the topological entropy of LTV systems. In general, the computation of the topological entropy of dynamical systems is known to be a difficult problem (see, e.g., Koiran, 2001, and Delvenne and Blondel, 2004). In fact, the exact value of the topological entropy of most widely-used systems (such as the Hénon map, the van Der Pol oscillator, etc.) is still unknown (see, e.g., Matveev and Pogromsky, 2016). On the other hand, many constructive ways of deriving lower bounds and upper bounds on the topological entropy of dynamical systems have been proposed in the literature (see, e.g., Liberzon and Mitra, 2018, Matveev and Pogromsky, 2016, and Kawan et al., 2021). In the case of switched linear systems with fixed switching signal, this question is thoroughly studied in Vicinansa and Liberzon (2019) and Yang et al. (2020), which provide lower bounds and upper bounds on the topological entropy of switched systems, depending on the matrices of the system and on the switching signal. In this thesis, we show how these bounds can sometimes be refined, using the framework of p-dominant switched linear systems.

In a second time, we study the question of the quantized observation and stabilization of switched linear systems under arbitrary switching, in the mode-dependent setting. In particular, we are interested in determining the minimal data rate required for state estimation and stabilization of switched linear systems (with affine control input), when the switching signal of the system is not fixed beforehand but is known at all times by all elements of the network; see also Figure 3.2 for an illustration.

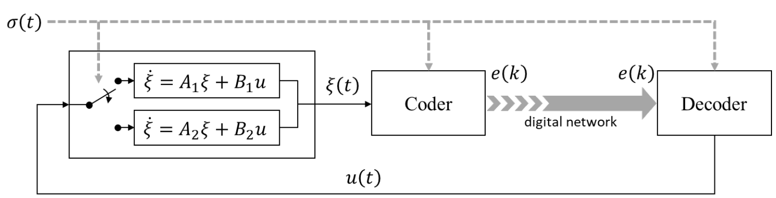


Figure 3.2: Mode-dependent quantized stabilization of switched linear systems.

This setting is motivated, for instance, by quantized control problems involving switched systems with exogenous switching signal that can be observed by all elements of the network, or switched systems with deterministic switching mechanism but for which the switching signal is not known at the time of communication infrastructure's design. This setting is also relevant for quantized control of switched linear systems with controlled switching signal, as the decoder controls both the mode and the affine input to achieve a given control objective; see also Example 3.6 in Subsection 3.3.4. Finally, this setting provides fundamental lower bounds and upper bounds on the data rate required in other settings, like mode-oblivious quantized control, or quantized control based on event-triggered communication strategies (see, e.g., Tallapragada and Cortés, 2016, and Pearson et al., 2017).

Mode-dependent quantized control of switched systems has been studied mainly in the context of *Markov Jump Linear Systems* (discrete-time control-affine switched linear systems whose sequence of modes is dictated by a Markov chain). Constructive data rate bounds for their Mean Square Stabilization have been proposed, e.g., in Zhang et al. (2009), Ling and Lin (2010) and Xiao et al. (2010), and an expression for the minimal data rate for Mean Square Stabilization, thought not computable in general, is derived in Nair et al. (2003).

Our contribution regarding the quantized control of switched linear systems in the mode-dependent setting is as follows. First, we introduce the concept of *worst-case topological entropy*, accounting for the maximal topological entropy that can be reached by the system among all its switching signals. We demonstrate the relevance of this concept by showing that the minimal data rate for state estimation or stabilization of a switched linear system corresponds to the worst-case topological entropy of the open-loop switched linear system. Moreover, we show that this optimal data rate bound can be approached by practical (i.e., implementable) coders–decoders. In particular, we describe the implementation of a practical coder–decoder that observes or stabilizes the system and whose data rate can be as close as desired to the worst-case topological entropy of the open-loop system. Secondly, we present a computable closed-form expression for the worst-case topological entropy of switched linear systems. The worst-case topological entropy is expressed as the joint spectral radius (see Definition 1.50 in Subsection 1.3.2) of some "lifted" system that represents the action of the original system on elements of volume (captured by algebraic constructions called *exterior algebras*). The main asset of this closed-form expression is that it can be computed numerically via well-established algorithms for the computation of the joint spectral radius (see, e.g., Jungers, 2009, and Sun and Ge, 2011); thereby allowing for practical computation of the minimal data rate for state estimation and stabilization of switched linear systems in the mode-dependent setting.

Mode-oblivious quantized control

Finally, we study the problem of quantized control of switched linear systems in the mode-oblivious setting. That is, we are interested in the state estimation and stabilization of switched linear systems (with affine control input), when both state and mode observation are subject to data-rate constraints; see also Figure 3.3 for an illustration.

When the mode of the system is only partially known by the decoder, state

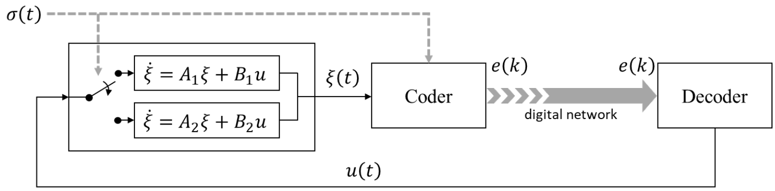


Figure 3.3: Mode-oblivious quantized stabilization of switched linear systems.

encoding and input actuating strategies must take into account the fact that unobserved switching may occur during the transmission interval. One way to limit the uncertainty on the switching signal between the transmission times is to impose *slow-switching conditions* on the switching signal to reduce its expressiveness. This technique was used in Liberzon (2014), Wakaiki and Yamamoto (2014) and Yang and Liberzon (2018), for the quantized stabilization of switched linear systems with dwell-time assumptions on the switching signal (see Definition 1.40 in Subsection 1.3.1). In particular, sufficient lower bounds on the data rate of the coder–decoder and on the absolute and average dwell time of the system are derived to ensure stabilization of the system. Entropy-related notions for non-autonomous dynamical systems with slow-varying inputs are also studied in the recent preprint by Sibai and Mitra (2020); in particular, a result (similar to Theorem 3.35 in this thesis) stating that these systems have in general a infinite topological entropy is presented.

Our main contribution regarding this setting is to push further the approach used in the above references and to study the necessity and sufficiency of the slow-switching conditions for the quantized stabilization of switched linear systems with mode-oblivious coder–decoder. More precisely, we show that switched linear systems under arbitrary switching are in general not stabilizable with a finite data rate. In particular, we present an example of switched linear system that is stabilizable for any switching signal in the absence of data-rate constraints, but cannot be stabilized with a finite data rate. This motivates the introduction of slow-switching assumptions in order to make the problem of quantized mode-oblivious stabilization of switched linear systems tractable. Then, we show that any stabilizable (without data-rate constraints) switched linear system with average dwell time bounded away from zero can be stabilized with a finite data rate. We present a sufficient upper bound on the data rate depending on the system and the average dwell time, and we describe the implementation of a coder–decoder that stabilizes the system. The difference with the above references is that the average dwell time can be arbitrarily close to zero, whereas in Liberzon (2014), Wakaiki and Yamamoto

(2014) and Yang and Liberzon (2018), the lower bound on the data rate does not converge to zero, even if the transmission frequency and the data rate tend to infinity. Finally, we also show that switched linear systems have in general an infinite topological entropy. This implies that these systems are in general *not observable* with arbitrary accuracy with a finite data rate, regardless of the slow-switching assumptions on the switching signal. This contrasts with the above result that switched linear systems with nonzero average dwell time are stabilizable with a finite data rate.

The results presented in this chapter have been reported in

- Guillaume O Berger and Raphaël M Jungers. Worst-case topological entropy and minimal data rate for state observation of switched linear systems. In *Proceedings of the 23rd International Conference on Hybrid Systems: Computation and Control*, pages 1–11. ACM, 2020e. doi: 10.1145/3365365.3382195.

- Guillaume O Berger and Raphaël M Jungers. Topological entropy and minimal data rate for state observation of LTV systems. *IFAC-PapersOnLine*, 53(2):3060–3065, 2020d. doi: 10.1016/j.ifacol.2020.12.1007.

- Guillaume O Berger and Raphaël M Jungers. Finite data-rate feedback stabilization of continuous-time switched linear systems with unknown switching signal. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 3823–3828. IEEE, 2020a. doi: 10.1109/CDC42340.2020.9304214.

- Guillaume O Berger and Raphaël M Jungers. Quantized stabilization of continuous-time switched linear systems. *IEEE Control Systems Letters*, 5(1):319–324, 2021c. doi: 10.1109/LCSYS.2020.3002068.

3.2 Quantized observation of linear time-varying systems

In this section, we study the minimal data rate for state estimation of linear time-varying systems. We show that this quantity coincides with the topological entropy of these systems, thereby extending the well-known "data rate theorems" for linear time-invariant systems and time-invariant nonlinear dynamical systems with compact domain. This result is relevant for the problem of observing linear time-varying systems over communication networks, as it provides a tight lower bound on the channel capacity required for the state estimation of these systems. This bound can be used for instance as a benchmark for the comparison of different implementations of coders–decoders observing the system.

Compared to time-invariant dynamical systems, the following difficulty arises when one seeks to relate the topological entropy with the minimal data rate for state estimation of time-varying systems. Since the coder–decoder communicates at periodic times, data rate requirements are mainly driven by the amount of information generated by the system *between two transmission times.* By contrast, the topological entropy gives the growth rate of information necessary to approximate the trajectories of the system *since the beginning of time.* For time-invariant systems, this problem is overcome by using the fact that the amount of information generated on each transmission interval is the same, and thus it is possible to relate the minimal data rate with the topological entropy. However, this is not true in general for time-varying systems. Therefore, in our analysis of linear time-varying systems, we have used a different approach exploiting the fact that from the growth rate of information generated by the system, i.e., the topological entropy, we can derive an upper bound on the amount of information generated during the sampling intervals. This requires that the trajectories of the system are spatially distributed in a "uniform way" (see Subsection 3.2.2 for details), and which is ensured by the linearity of the system; in particular, it seems that the proof argument presented in this paper does not extend straightforwardly to nonlinear time-varying systems.

The section is organized as follows. In Subsection 3.2.1, we remind the notions of linear time-varying systems, topological entropy and minimal data rate for state estimation of these systems, and we introduce the problem of interest. In Subsection 3.2.2, we present and prove the main result of this section, namely the equivalence of the topological entropy and the minimal data rate for state estimation of linear time-varying systems. Finally, in Subsection 3.2.3, we discuss the applications of the framework of p-dominance of switched linear systems (introduced in Section 2.2) for the computation of the topological entropy of linear time-varying systems.

Notation. We will consider both continuous-time systems and discrete-time systems, and we will use the symbol $\mathbb{T}$ to denote the time domain of the system, as it should be clear from the context whether $\mathbb{T} = \mathbb{R}$ (continuous-time systems) or $\mathbb{T} = \mathbb{Z}$ (discrete-time systems). The restriction of a function $f : A \to B$ to a set $A' \subseteq A$ is denoted by $f|_{A'}$. The Minkowski sum of $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$ (or $\{x\} \subseteq \mathbb{R}^n$) is denoted by $A + B$ (or $A + x$). $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ are the ceil and floor functions.

3.2.1 Problem setting

We consider Linear Time-Varying (LTV) systems, which are time-varying dynamical systems[1] of the form $(\mathbb{R}^n, f)$ where $f(t,x) = \hat{A}(t)x$ for all $t \in \mathbb{T}_{\geq 0}$ and $x \in \mathbb{R}^n$, and $\hat{A} : \mathbb{T}_{\geq 0} \to \mathbb{R}^{n\times n}$ is continuous. Let $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$ be a LTV system. For a reminder, the trajectories $\xi : \mathbb{T}_{\geq 0} \to \mathbb{R}^n$ of Sys satisfy that $\dot{\xi}(t) = \hat{A}(t)\xi(t)$ for all $t \in \mathbb{R}_{\geq 0}$ (if Sys is continuous-time) and $\xi(t+1) = \hat{A}(t)\xi(t)$ for all $t \in \mathbb{N}$ (if Sys is discrete-time). Following the notation of switched linear systems, we let $\chi(\cdot,\cdot,\cdot;\mathsf{Sys})$ (or simply χ when Sys is clear from the context) be the *generator*[2] and $\mathring{\chi}(\cdot,\cdot;\mathsf{Sys})$ (or simply $\mathring{\chi}$ when Sys is clear from the context) be the *fundamental matrix solution*[3] of Sys.

Topological entropy of LTV systems

The following definition of topological entropy particularizes the one for hybrid systems (see Definition 1.78 in Subsection 1.5.2) to LTV systems.

Definition 3.1 (Topological entropy of a LTV system)**.** *Consider a LTV system* $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$ *and a bounded set* $X_0 \subseteq \mathbb{R}^n$*. The* topological entropy *of* Sys *starting from* X_0*, denoted by* $h_{\text{top}}(\mathsf{Sys}, X_0)$*, is defined as the topological entropy of the hybrid system associated to* Sys*, with initial set* X_0 *for the variable "x", and with respect to the cost function* $\mathfrak{C} : (\mathbb{R}\times\mathbb{R}^n)\times(\mathbb{R}\times\mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ *defined by* $\mathfrak{C}(t_1,x_1,t_2,x_2) = \|x_1 - x_2\|$*.*

In other words, $h_{\text{top}}(\mathsf{Sys}, X_0)$ in Definition 3.1 is defined as

$$h_{\text{top}}(\mathsf{Sys}, X_0) = \sup_{\epsilon>0} \limsup_{T\to\infty} \frac{1}{T} \log_2 s_{\text{span}}(\epsilon, T; X_0), \tag{3.1}$$

where $s_{\text{span}}(\epsilon, T; X_0)$ is the smallest cardinality of an (ϵ, T)-spanning set for Sys starting from X_0, that is, the minimal number of functions necessary to approximate, with accuracy ϵ on the interval $[0,T) \cap \mathbb{T}$, all trajectories of Sys starting in X_0 (see Definition 1.76 in Subsection 1.5.2 for details). Equivalently, $h_{\text{top}}(\mathsf{Sys}, X_0)$ can be defined as

$$h_{\text{top}}(\mathsf{Sys}, X_0) = \sup_{\epsilon>0} \limsup_{T\to\infty} \frac{1}{T} \log_2 s_{\text{sep}}(\epsilon, T; X_0), \tag{3.2}$$

where $s_{\text{sep}}(\epsilon, T; X_0)$ is the largest cardinality of an (ϵ, T)-separated set for Sys starting from X_0, that is, the maximal number of trajectories of Sys starting in

[1] We refer the reader to Subsection 1.1.2 for the definition of time-varying dynamical systems.

[2] See Definition 1.36 in Subsection 1.3 for the definition in the case of switched linear systems.

[3] See Definition 1.47 in Subsection 1.3 for the definition in the case of switched linear systems.

X_0 that are ϵ-distinguishable on $[0, T) \cap \mathbb{T}$ (see Definition 1.77 and Proposition 1.79 in Subsection 1.5.2 for details).

It follows from the linearity of LTV systems that the topological entropy of these systems is *independent of the initial set* X_0, as long as it is bounded with nonempty interior; the reason is that the quantity s_{span} (or s_{sep}) is invariant by spatial translation of the initial set (so that we can assume that 0 is in the interior of X_0), and from the scaling invariance of linear systems, it holds that for any $c > 0$, $s_{\mathrm{span}}(c\epsilon, T; cX_0) = s_{\mathrm{span}}(\epsilon, T; X_0)$ (see, e.g., Yang et al., 2020, Proposition 2 for details). For this reason, in the following, we will omit the initial set in the notation, and simply use $h_{\mathrm{top}}(\mathsf{Sys})$ to denote the topological entropy of the LTV system Sys starting from any bounded initial set with nonempty interior.

The example below illustrates the notions of spanning sets, separated sets, and topological entropy, with a simple LTV system.

Example 3.1. Consider the discrete-time LTV system $\mathsf{Sys} = (\mathbb{R}^1, \hat{A})$ where $\hat{A} : \mathbb{N} \to \mathbb{R}^{1\times 1}$ is defined by $\hat{A}(t) = 1$ if t is even and $\hat{A}(t) = 2$ if t is odd. The generator of Sys is thus given by $\chi(t, x) = 2^{\lfloor t/2 \rfloor} x$ for all $t \in \mathbb{N}$. We will show that $h_{\mathrm{top}}(\mathsf{Sys}) = 1/2$. As explained above, the topological entropy of LTV systems does not depend on the initial set; hence, we set $X_0 = [0, 1]$.

<u>Step 1:</u> We show that $h_{\mathrm{top}}(\mathsf{Sys}) \leq 1/2$, by using (3.1). Therefore, fix $\epsilon > 0$ and $T \in \mathbb{N}$, and let $N = \lceil \epsilon^{-1} 2^{(T-3)/2} \rceil$. Define $E = \{0, 1/N, 2/N, \ldots, 1\}$ and let $\mathcal{E} = \{\chi(\cdot, x)|_{\{0,\ldots,T-1\}}\}_{x\in E}$. We show that $\mathcal{E}$ is (ϵ, T)-spanning for Sys starting from X_0. Therefore, fix $x \in X_0$, and let $\hat{x} \in E$ minimize the distance to x. Then, by definition of E, it holds that $|x - \hat{x}| \leq 1/(2N) \leq \epsilon 2^{(1-T)/2}$. This implies that $|\chi(t, x) - \chi(t, \hat{x})| = 2^{\lfloor t/2 \rfloor} |x - \hat{x}| \leq \epsilon$ for all $t \in \{0, \ldots, T-1\}$. Thus, $\mathcal{E}$ is (ϵ, T)-spanning for Sys starting from X_0, so that $s_{\mathrm{span}}(\epsilon, T; X_0) \leq |E| = \lceil \epsilon^{-1} 2^{(T-3)/2} \rceil + 1$. Since ϵ and T were arbitrary, it follows that $h_{\mathrm{top}}(\mathsf{Sys})$ is upper-bounded by $\sup_{\epsilon>0} \limsup_{T\to\infty} \frac{1}{T} \log_2(\lceil \epsilon^{-1} 2^{(T-3)/2} \rceil + 1) = 1/2$.

<u>Step 2:</u> We show that $h_{\mathrm{top}}(\mathsf{Sys}) \geq 1/2$, by using (3.2). Therefore, fix $\epsilon > 0$ and $T \in \mathbb{N}$ odd. Let $N = \lceil \epsilon^{-1} 2^{(T-1)/2} \rceil - 1$. We assume that $N > 0$ since in (3.2) we take the limit when $\epsilon \to 0$ and $T \to \infty$. Define $F = \{0, 1/N, 2/N, \ldots, 1\}$. Then, any distinct $x_1, x_2 \in F$ satisfy that $|x_1 - x_2| > \epsilon 2^{(1-T)/2}$, so that $|\chi(T-1, x_1) - \chi(T-1, x_2)| > \epsilon$. Thus, the set $\{\chi(\cdot, 0, x)|_{\{0,\ldots,T-1\}}\}_{x\in F}$ is (ϵ, T)-separated for Sys starting from X_0, implying that $s_{\mathrm{sep}}(\epsilon, T; X_0) \geq |F| = \lceil \epsilon^{-1} 2^{(T-1)/2} \rceil$. Since ϵ and T were arbitrary, and by injecting in (3.2), we get that $h_{\mathrm{top}}(\mathsf{Sys}) \geq 1/2$. This concludes the example.

Minimal data rate for state estimation of LTV systems

The following definition of minimal data rate for state estimation of LTV systems particularizes the one for hybrid systems (see Definition 1.74 in Subsection 1.5.2) in the case of LTV systems.

Definition 3.2 (Minimal data rate for state estimation of a LTV system)**.** *Consider a LTV system* $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$ *and a bounded set* $X_0 \subseteq \mathbb{R}^n$*. The* minimal data rate for state estimation *of* Sys *starting from* X_0*, denoted by* $\mathcal{R}_{\mathrm{est}}(\mathsf{Sys}, X_0)$*, is defined as the minimal data rate for state estimation of the hybrid system associated to* Sys*, with initial set* X_0 *for the variable "x", and with respect to the cost function* $\mathfrak{C} : (\mathbb{R} \times \mathbb{R}^n) \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ *defined by* $\mathfrak{C}(t_1, x_1, t_2, x_2) = \|x_1 - x_2\|$*.*

In other words, $\mathcal{R}_{\mathrm{est}}(\mathsf{Sys}, X_0)$ in Definition 3.2 is defined as

$$\mathcal{R}_{\mathrm{est}}(\mathsf{Sys}, X_0) = \sup_{\epsilon > 0} \inf_{\mathsf{CoDec}} \mathcal{R}(\mathsf{CoDec}),$$

where the infimum is over all coders–decoders CoDec that ϵ-observe Sys starting from X_0, and where $\mathcal{R}(\mathsf{CoDec})$ is the data rate of CoDec (see Definitions 1.71 and 1.73 in Subsection 1.5.1). Following the definitions in Subsection 1.5.1, a coder–decoder for Sys starting from X_0, with transmission period T_{t}, can be described as an ordered pair $((\Psi^{\mathrm{c}}_k)_{k\in\mathbb{N}}, (\Psi^{\mathrm{d}}_k)_{k\in\mathbb{N}})$ where for every $k \in \mathbb{N}$, $\Psi^{\mathrm{c}}_k : X_0 \to Y_{\mathrm{t}}$ is the *coder function* at step k and $\Psi^{\mathrm{d}}_k : (Y_{\mathrm{t}})^{k+1} \times [kT_{\mathrm{t}}, (k+1)T_{\mathrm{t}}) \cap \mathbb{T} \to X$ is the *decoder function* at step k. At each time $t = kT_{\mathrm{t}}$, $k \in \mathbb{N}$, the coder outputs a symbol $e(k)$ defined by $e(k) = \Psi^{\mathrm{c}}_k(x)$ where x is the initial condition of the system. The symbols are transmitted to the decoder, which produces at each time $t \in [kT_{\mathrm{t}}, (k+1)T_{\mathrm{t}}) \cap \mathbb{T}$, $k \in \mathbb{N}$, an estimate $\hat{\xi}(t) = \Psi^{\mathrm{d}}_k(e(0), \ldots, e(k), t)$ of the current state of the system. The coder–decoder $\mathsf{CoDec} = ((\Psi^{\mathrm{c}}_k)_{k\in\mathbb{N}}, (\Psi^{\mathrm{d}}_k)_{k\in\mathbb{N}})$ is said to *ϵ-observe* Sys starting from X_0 if for every trajectory $\xi : \mathbb{T}_{\geq 0} \to \mathbb{R}^n$ of Sys with $\xi(0) \in X_0$ and every $t \in \mathbb{T}_{\geq 0}$, $\|\xi(t) - \hat{\xi}(t)\| \leq \epsilon$ where $\hat{\xi}(t)$ is defined as above, with $x = \xi(0)$. Finally, the data rate of CoDec is defined as the maximal number of bits per unit of time necessary to encode the symbols: $\mathcal{R}(\mathsf{CoDec}) = \frac{\lceil \log_2 |Y_{\mathrm{t}}| \rceil}{T_{\mathrm{t}}}$.

For linear time-invariant systems and hybrid systems whose initial set is forward invariant, it is well known that the topological entropy and the minimal data rate for state estimation coincide (see Proposition 1.80 in Subsection 1.5.2). This result—sometimes referred to as a *data-rate theorem* (see, e.g., Kawan et al., 2021, p. 1)—does not apply in general to time-varying dynamical systems because the initial set of these systems is not forward invariant (see Definitions 1.9 and 1.10 in Subsection 1.1.2 for the definitions of time-varying dynamical systems as hybrid systems). However, we will see in the next subsection that, for the special case of LTV systems, the data-rate theorem does hold.

3.2.2 Equivalence of topological entropy and minimal data rate for LTV systems

The following theorem is the main result of this section. It extends the data-rate theorem (Proposition 1.80 in Subsection 1.5.2) to LTV systems.

Theorem 3.3. *Consider a LTV system* $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$ *and a bounded set* $X_0 \subseteq \mathbb{R}^n$ *with nonempty interior. It holds that* $h_{\mathrm{top}}(\mathsf{Sys}) = \mathcal{R}_{\mathrm{est}}(\mathsf{Sys}, X_0)$.

The proof of Theorem 3.3 is presented in the subsubsection below. Then, in the afternext subsubsection, we discuss the consequences and the practical applicability of the theorem.

Proof of Theorem 3.3

The fact that $\mathcal{R}_{\mathrm{est}}(\mathsf{Sys}, X_0) \geq h_{\mathrm{top}}(\mathsf{Sys})$ in Theorem 3.3 was already proved in Proposition 1.80. The rest of this subsubsection is thus devoted to the proof of $\mathcal{R}_{\mathrm{est}}(\mathsf{Sys}, X_0) \leq h_{\mathrm{top}}(\mathsf{Sys})$. Therefore, we first introduce a few definitions and notation.

Consider a LTV system $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$ and $T \in \mathbb{T}_{>0} \cup \{\infty\}$. We define the function $\|\cdot\|_{\mathsf{Sys},T} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by

$$\|x\|_{\mathsf{Sys},T} = \sup_{t \in [0,T) \cap \mathbb{T}} \|\mathring{\chi}(t,0)x\| \qquad \text{for all } x \in \mathbb{R}^n,$$

where $\mathring{\chi}$ is the fundamental matrix solution of Sys. It holds that $\|\cdot\|_{\mathsf{Sys},T}$ is a *norm* on $\mathbb{R}^n$; indeed, it satisfies that the axioms of positive definiteness (since $\|x\|_{\mathsf{Sys},T} \geq \|x\|$), triangular inequality and positive homogeneity (as the supremum of a set of functions satisfying each the triangular inequality and the positive homogeneity). Note that by the definition of $\|\cdot\|_{\mathsf{Sys},T}$, it holds that if $x_1, x_2 \in \mathbb{R}^n$ satisfy $\|x_1 - x_2\|_{\mathsf{Sys},T} > \epsilon$, then there is $t \in [0,T) \cap \mathbb{T}$ such that $\|\chi(t,0,x_1) - \chi(t,0,x_2)\| > \epsilon$, so that the trajectories of Sys starting from x_1 and x_2 respectively are (ϵ, T)-separated. Hence, the norm $\|\cdot\|_{\mathsf{Sys},T}$ allows us to provide an equivalent definition of (ϵ, T)-separated sets in the case of LTV systems; this will be used in Lemma 3.5 below. Finally, for every $a \in \mathbb{R}^n$ and $r \geq 0$, we let $\mathbb{B}_{\mathsf{Sys},T}(a,r) = \{x \in \mathbb{R}^n : \|x - a\|_{\mathsf{Sys},T} \leq r\}$. Note that, by the positive homogeneity of norms and the definition of $\mathbb{B}_{\mathsf{Sys},T}$, it holds that for any $c > 0$, $\mathbb{B}_{\mathsf{Sys},T}(a, cr) = a + c\mathbb{B}_{\mathsf{Sys},T}(0, r)$.

Definition 3.4. *Let $\epsilon > 0$, $\Lambda \subseteq \mathbb{R}^n$ and $E \subseteq \mathbb{R}^n$. We say that $E \subseteq \mathbb{R}^n$ is an* $(\epsilon, T; \mathsf{Sys})$-cover *(or* (ϵ, T)-cover *if* Sys *is clear from the context) of Λ if $\Lambda \subseteq \bigcup_{a\in E} \mathbb{B}_{\mathsf{Sys},T}(a, \epsilon)$. The smallest cardinality of an $(\epsilon, T; \mathsf{Sys})$-cover of Λ is denoted by $s_{\mathrm{cov}}(\epsilon, T; \mathsf{Sys}, \Lambda)$ (or $s_{\mathrm{cov}}(\epsilon, T; \Lambda)$ if Sys is clear from the context). An $(\epsilon, T; \mathsf{Sys})$-cover of Λ with cardinality equal to $s_{\mathrm{cov}}(\epsilon, T; \mathsf{Sys}, \Lambda)$ is said to be* minimal*.*

Idea of the proof: The idea behind the proof of $\mathcal{R}_{\mathrm{est}}(\mathsf{Sys}, X_0) \leq h_{\mathrm{top}}(\mathsf{Sys})$ in Theorem 3.3 is that for a LTV system $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$ and a set $\Lambda \subseteq \mathbb{R}^n$, any minimal (ϵ, T)-cover of Λ must be "uniformly distributed" over Λ. From the uniform distribution, we get that if E_1 is a minimal (ϵ, T_1)-cover of Λ and E_2 is an (ϵ, T_2)-cover of Λ, with $T_2 \geq T_1$, then there is an (ϵ, T_2)-cover of $\mathbb{B}_{\mathsf{Sys},T_1}(0, \epsilon)$ with cardinality of the order of $|E_2|/|E_1|$. These claims are encapsulated in the following lemmas.[4]

Lemma 3.5. *Consider a LTV system $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$ and a bounded set $\Lambda \subseteq \mathbb{R}^n$. Let $\epsilon > 0$ and $T \in \mathbb{T}_{>0}$, and let $F \subseteq \Lambda$ be a set with maximal cardinality such that $\{\chi(\cdot, x)|_{[0,T)\cap\mathbb{T}}\}_{x\in F}$ is (ϵ, T)-separated for Sys starting from Λ. Then, it holds that F is an (ϵ, T)-cover of Λ.*

Proof. For every $y \in \Lambda$, there is $\hat{x} \in F$ such that $\|y - \hat{x}\|_{\mathsf{Sys},T} \leq \epsilon$, as otherwise $F \cup \{y\} \subseteq \Lambda$ would satisfy that $\{\chi(\cdot, x)|_{[0,T)\cap\mathbb{T}}\}_{x\in F\cup\{y\}}$ is an (ϵ, T)-separated set for Sys, contradicting the maximality of F. This implies that F is an (ϵ, T)-cover of Λ, concluding the proof. □

Lemma 3.6. *Consider a LTV system $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$, and let $\epsilon > 0$ and $T \in \mathbb{T}_{>0}$. Then, for every $a \in \mathbb{R}^n$ and $c \geq 0$, there is an (ϵ, T)-cover E of $\mathbb{B}_{\mathsf{Sys},T}(a, c\epsilon)$ with $|E| \leq (2c + 1)^n$.*

Proof. See Appendix A.3.1. □

Lemma 3.7. *Consider a LTV system $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$ and a bounded set $\Lambda \subseteq \mathbb{R}^n$. Let $\epsilon > 0$ and $T_1, T_2 \in \mathbb{T}_{>0}$, $T_2 \geq T_1$. Let E_1 be a minimal (ϵ, T_1)-cover of Λ and let E_2 be an (ϵ, T_2)-cover of $\Lambda + \mathbb{B}_{\mathsf{Sys},T_1}(0, 2\epsilon)$. Then, there exists an (ϵ, T_2)-cover E of $\mathbb{B}_{\mathsf{Sys},T_1}(0, \epsilon)$ with cardinality $|E| \leq 11^n|E_2|/|E_1|$.*

Proof. See Appendix A.3.2. □

We are now able to prove that $\mathcal{R}_{\mathrm{est}}(\mathsf{Sys}, X_0) \leq h_{\mathrm{top}}(\mathsf{Sys})$ in Theorem 3.3.

[4] The proofs of the first two lemmas connect with the concepts of *packing* and *covering* in machine learning; see, e.g., Shalev-Shwartz and Ben-David (2014, Chapter 27). See also, e.g., Wang et al. (2021) for applications in data-driven control.

Proof of $\mathcal{R}_{\text{est}}(\mathsf{Sys}, X_0) \leq h_{\text{top}}(\mathsf{Sys})$ *in Theorem 3.3.* Since X_0 is bounded and since $\mathcal{R}_{\text{est}}(\mathsf{Sys}, X_0)$ is increasing with X_0, we may assume without loss of general that X_0 is the Euclidean ball centered at $0 \in \mathbb{R}^n$ and with radius $r \geq 1$. We will show that $h_{\text{top}}(\mathsf{Sys}) = \mathcal{R}_{\text{est}}(\mathsf{Sys}, X_0)$.

From Lemma 3.5, it holds that $s_{\text{cov}}(\epsilon, T; X_0) \leq s_{\text{sep}}(\epsilon, T; X_0)$. Hence, by (3.2), we get that

$$\sup_{\epsilon>0} \limsup_{T\to\infty} \frac{1}{T} \log_2 s_{\text{cov}}(\epsilon, T; X_0) \leq h_{\text{top}}(\mathsf{Sys}). \tag{3.3}$$

Let $R > h_{\text{top}}(\mathsf{Sys})$ and $\epsilon \in (0, 1/2)$. We build a coder–decoder that ϵ-observes Sys and whose data rate is smaller than R. For the sake of simplicity, we assume that Sys is continuous-time; the proof with Sys discrete-time is along the same lines and can be found in Berger and Jungers (2020d, Section 3).

To build such a coder–decoder, fix $\alpha \in \mathbb{R}$ such that $h_{\text{top}}(\mathsf{Sys}) < \alpha < R$, and then, using (3.3), let $T \in \mathbb{R}_{>0}$ be large enough so that $\lfloor RT \rfloor \geq \alpha T + 8n$, and for all $T' \in [T, \infty)$, $s_{\text{cov}}(\epsilon, T'; X_0) \leq 2^{\alpha T'}$. This T will be the transmission period of the coder–decoder. The implementation of the coder and the decoder is given in Figure 3.4.

We prove that the coder–decoder described in Figure 3.4 ϵ-observes Sys. Therefore, let ξ be a trajectory of Sys starting in X_0. Fix $k \in \mathbb{N}$, and let T_k and E_k be defined as in the implementation of the coder–decoder. By definition of $\hat{x}_k$, it holds that $\xi(0) \in \mathbb{B}_{\mathsf{Sys},T_k}(\hat{x}_k, \epsilon)$, which means that for all $t \in [0, T_k]$, $\|\xi(t) - \chi(t, \hat{x}_k)\| \leq \epsilon$. Moreover, by definition of T and T_k, it is clear $T_k \geq (k+1)T$. Hence, it follows by definition of $\hat{\xi}$ that for all $t \in [kT, (k+1)T)$, $\|\xi(t) - \hat{\xi}(t)\| \leq \epsilon$. Since k was arbitrary, this shows that the coder–decoder ϵ-observes Sys.

Finally, it remains to show that the data rate of the coder–decoder is smaller than R, and this is where we will use the results (Lemmas 3.6 and 3.7) on the "uniform distribution" of (ϵ, T)-covers of X_0. Therefore, fix $k \in \mathbb{N}$, and let T_k and E_k be defined as in the implementation of the coder. Using Lemmas 3.6 and 3.7, we will show that $|E_k| \leq 220^n 2^{\alpha T}$ (see Lemma 3.9 below). First, we will need the following lemma.

Lemma 3.8. *Consider a LTV system* $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$ *and a set* $\Lambda \subseteq \mathbb{R}^n$. *Let* $\epsilon > 0$ *and* $T \in \mathbb{R}_{>0}$. *There is* $T_* \in \mathbb{R}$, $T_* > T$, *such that* $s_{\text{cov}}(\epsilon, T_*; \Lambda) \leq 4^n s_{\text{cov}}(\epsilon, T; \Lambda)$.

Proof. See Appendix A.3.3. □

Lemma 3.9. *Let* α, T, $\{T_k\}_{k\in\mathbb{N}}$ *and* $\{E_k\}_{k\in\mathbb{N}}$ *be as in the implementation of the coder. It holds that for all* $k \in \mathbb{N}$, $|E_k| \leq 220^n 2^{\alpha T}$.

Proof. The claim is clear for $k = 0$, by definition of T_k. Now, assume $k \in \mathbb{N}_{>0}$. Since $\epsilon < r/2$, it holds that $X_0 + \mathbb{B}_{\mathsf{Sys},T_k}(0, 2\epsilon) \subseteq 2X_0$. By Lemma 3.6, it is possible to cover $2X_0$ with 5^n translated copies of X_0, so that $s_{\text{cov}}(\epsilon, T_k; 2X_0) \leq 5^n s_{\text{cov}}(\epsilon, T_k; X_0)$. Thus, by using Lemma 3.7, we get that $|E_k| \leq 55^n s_{\text{cov}}(\epsilon, T_k; X_0)/ s_{\text{cov}}(\epsilon, T_{k-1}; X_0)$. Moreover, by definition of T_{k-1}, it holds that $s_{\text{cov}}(\epsilon, T_{k-1}; \Lambda) \geq 4^{-n} 2^{\alpha k T}$ (see Lemma 3.8). Hence, we have that $|E_k| \leq (55^n 2^{\alpha(k+1)T})/(4^{-n} 2^{\alpha k T}) = 220^n 2^{\alpha T}$. □

Since $\lfloor RT \rfloor \geq \alpha T + 8n$, it is sufficient to have $\lfloor RT \rfloor$ bits to give a unique number to each of the $220^n 2^{\alpha T}$ points of E_k, since $\log_2(220^n 2^{\alpha T}) \leq \alpha r + 8n$. Hence, the symbol $e(k)$ in the definition of the coder–decoder can be encoded with $\lfloor RT \rfloor$ bits. This shows that the data rate of the coder–decoder is at most equal to $\lfloor RT \rfloor / T \leq R$, which concludes the proof of Theorem 3.3. □

Discussion of Theorem 3.3

It should be noted that the result of Theorem 3.3 is rather theoretical, since the practical implementation of the coder–decoder described in Figure 3.4 can be quite intricate. In fact, the challenging part of the implementation is not so much the computation of the minimal (ϵ, T_k)-covers E_k (for $k \in \mathbb{N}$), which can be achieved by computing minimal-volume ellipsoidal enclosures of the balls $\mathbb{B}_{\mathsf{Sys},T_k}$ (see, e.g., Boyd and Vandenberghe, 2004, Section 8.4), but rather the computation of the times T_k (for $k \in \mathbb{N}$), which supposes that the values of $\hat{A}$ are known over a potentially infinite time horizon. This assumption is not always satisfiable in practice; in particular, it may imply that the memory of the coder–decoder is infinite. This assumption is however crucial in the proof of the theorem, as it ensures that T_k can be chosen so that $s_{\text{cov}}(\epsilon, T_{k-1}; \Lambda)$ is bounded from below (see the proof of Lemma 3.9). We nevertheless believe that Theorem 3.3 can be useful for practical purposes, as it gives a fundamental lower bound on the minimal data rate for state estimation of LTV systems, which can be used as a benchmark to compare the efficiency, in terms of communication resources, of different coders–decoders observing a given LTV system.

Finally, let us mention that the proof argument used in the proof of Theorem 3.3 relies on the linearity of the system, which ensures that the minimal covers of the initial set are "uniformly distributed" (see Lemma 3.7 for a formal statement). In particular, it is not straightforward how to generalize this argument to nonlinear time-varying dynamical systems, or dynamical systems whose initial set is not forward invariant. On the other hand, we do not have a counter-example of such a system for which the topological entropy and the minimal data rate for state estimation do not coincide. There seems thus to be a gap in the theory, which could be an interesting direction for further research.

3.2.3 Computation of the topological entropy of LTV systems

In this subsection, we show that the theory of p-dominance (introduced in Section 2.2) can be used to compute bounds on the topological entropy of LTV systems obtained from p-dominant switched linear systems. More precisely, given a switched linear system $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ and a switching signal $\sigma : \mathbb{T}_{\geq 0} \to \Sigma$, we consider the *associated LTV system* $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$ where $\hat{A} : \mathbb{T}_{\geq 0} \to \Sigma$ is defined by $\hat{A}(t) = A_{\sigma(t)}$ for all $t \in \mathbb{T}_{\geq 0}$. We denote by $h_{\mathrm{top}}(\mathsf{SwS}; \sigma)$ the topological entropy of Sys. Our goal is to give lower bounds and upper bounds on $h_{\mathrm{top}}(\mathsf{SwS}; \sigma)$ when SwS is p-dominant and the rates of p-dominance are known (see Definition 2.8 in Subsection 2.2.2).

For the sake of simplicity, we restrict our attention to *discrete-time* LTV systems obtained from switched linear systems. However, using the theory of p-dominance for dynamical systems (introduced in Section 2.3), one can easily extend the results presented in this subsection to all discrete-time LTV systems, and to continuous-time LTV systems by using time-discretization.

Before presenting these results, let us mention that the question of estimating the topological entropy of switched linear systems with fixed switching signal was thoroughly studied in Yang et al. (2020) for general switched linear systems, and in Vicinansa and Liberzon (2019) for switched linear systems with a "*regular*" switching signal. The first reference provides the following lower bound and upper bound on the topological entropy of switched linear systems with fixed switching signal (the statement of the theorem has been adapted to discrete-time systems).

Proposition 3.10 (Yang et al., 2020, Theorem 1)**.** *Consider a discrete-time switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *and a switching signal* $\sigma : \mathbb{N} \to \Sigma$. *Let* $\|\cdot\|_*$ *be any sub-multiplicative matrix norm. For each* $T \in \mathbb{N}$ *and* $i \in \Sigma$, *let* $\rho_i(T) = \frac{1}{T}|\{t \in [0, T) \cap \mathbb{N} : \sigma(t) = i\}|$ *be the portion of time spent by the system in mode "i" on the time interval* $\{0, \ldots, T-1\}$. *Then, it holds that*

$$\max\left\{\limsup_{T\to\infty} \sum_{i\in\Sigma} \rho_i(T)\log_2|\det(A_i)|, 0\right\} \leq h_{\mathrm{top}}(\mathsf{SwS}; \sigma) \leq \max\left\{\limsup_{T\to\infty} n\sum_{i\in\Sigma} \rho_i(T)\log_2\|A_i\|_*, 0\right\}.$$

The above bounds can sometimes be very conservative: for instance, if one of the matrices of the system has a zero eigenvalue, then the lower bound in

Proposition 3.10 is equal to zero; on the other hand, if all matrices have a common eigenvector associated to a very large eigenvalue λ_* and have zero eigenvalues otherwise, then the upper bound will be equal to $n \log_2 \lambda_*$ while it should be equal to $\log_2 \lambda_*$. It turns out that the above bounds can be refined when the switched linear system is p-dominant with $0 < p < n$ and the rates of p-dominance are known.

Theorem 3.11. *Consider a discrete-time switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *and a switching signal* $\sigma : \mathbb{N} \to \Sigma$. *Let* $\|\cdot\|_*$ *be any sub-multiplicative matrix norm. Assume that* SwS *is* p*-dominant with automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$ *and set of rates* $\{\gamma_\theta\}_{\theta\in\Theta} \subseteq \mathbb{R}_{>0}$. *Let* $(\theta_t)_{t=0}^{\infty} \subseteq \Theta$ *be a path in* Aut *such that* $\sigma(t) = \mathsf{i}(\theta_t)$ *for all* $t \in \mathbb{N}$. *It holds that*

$$\max\{p\hat{\gamma}, 0\} \leq h_{\mathrm{top}}(\mathsf{SwS}; \sigma) \leq \\ \max\{(n-p)\hat{\gamma}, 0\} + \max\Big\{\limsup_{T\to\infty} p \sum_{i\in\Sigma} \rho_i(T) \log_2 \|A_i\|_*, 0\Big\},$$

where $\hat{\gamma} = \limsup_{T\to\infty} \frac{1}{T} \log_2(\gamma_{\theta_0} \cdots \gamma_{\theta_{T-1}})$, *and* $\rho_i(T)$, *for* $i \in \Sigma$ *and* $T \in \mathbb{N}$, *is as in Proposition 3.10.*

Proof. See Appendix A.3.4. □

Corollary 3.12. *Consider a discrete-time switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *under arbitrary switching. Assume that* SwS *is* p*-dominant with automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$ *and set of rates* $\{\gamma_\theta\}_{\theta\in\Theta} \subseteq \mathbb{R}_{>0}$, *and assume that* Aut *is cycle-stable with respect to* $\{\gamma_\theta\}_{\theta\in\Theta}$. *Then, for any switching signal* $\sigma \in \mathcal{S}$, *it holds that*

$$h_{\mathrm{top}}(\mathsf{SwS}; \sigma) \leq \max\{p\hat{\varrho}(\mathsf{SwS}), 0\},$$

where $\hat{\varrho}(\mathsf{SwS})$ *is the joint spectral radius of* SwS *(see Definition 1.50 in Subsection 1.3.2).*

Proof. Straightforward for Theorem 3.11 and the definition of cycle-stable automaton (see Definition 2.14 in Subsection 2.2.2). □

An example of application of Theorem 3.11 for the computation of the topological entropy of a p-dominant discrete-time switched linear system, and comparison with the bound in Proposition 3.10, is presented in Subsection 2.2.4.

3.3 Quantized control of switched linear systems with mode-dependent coder–decoder

In this section, we study the problem of state estimation and stabilization of switched linear systems with a mode-dependent coder–decoder. We introduce the concept of worst-case topological entropy of a switched linear system, and we show that the minimal data rate for state estimation or stabilization of switched linear systems coincides with the worst-case topological entropy of the open-loop system. We also derive a closed-form expression for the worst-case topological entropy, expressed as the joint spectral radius (see Definition 1.50 in Subsection 1.3.2) of some lifted switched linear system obtained from the original one by using tools from multilinear algebra. Drawing on this expression, we describe a practical coder–decoder that observes or stabilizes the system, and whose data rate can be as close as desired to the optimal data rate.

The section is organized as follows. In Subsection 3.3.1, we introduce the concept of worst-case topological entropy of a switched linear system. We also introduce the notions of minimal data rate for state estimation and stabilization of switched linear systems with a mode-dependent coder–decoder. In Subsection 3.3.2, we present the closed-form expression for the worst-case topological entropy and discuss the computability aspects. In Subsection 3.3.3, we demonstrate the equivalence of the worst-case topological entropy and the minimal data rate for state estimation and stabilization of switched linear systems, and we describe a practical coder–decoder that observes or stabilizes such systems and whose data rate can be as close as desired to the worst-case topological entropy. Finally, in Subsection 3.3.4, we demonstrate the applicability of our results on numerical examples.

Notation. We will consider both continuous-time systems and discrete-time systems, and we will use the symbol $\mathbb{T}$ to denote the time domain of the system, as it should be clear from the context whether $\mathbb{T} = \mathbb{R}$ (continuous-time systems) or $\mathbb{T} = \mathbb{Z}$ (discrete-time systems). The restriction of a function $f : A \to B$ to a set $A' \subseteq A$ is denoted by $f|_{A'}$. The Minkowski sum of $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$ (or $\{x\} \subseteq \mathbb{R}^n$) is denoted by $A + B$ (or $A + x$). If $A \subseteq \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$, then MA is the image of A by M, i.e., $MA = \{Mx : x \in A\}$. $\lceil\cdot\rceil$, $\lfloor\cdot\rfloor$ and $[\![\cdot]\!]$ are the ceil, floor and round functions.

3.3.1 Problem setting

We introduce the problem of interest of this section: namely, the study of the minimal data rate for state estimation and stabilization of switched linear systems with a mode-dependent coder–decoder, and its connection with the concept of *worst-case topological entropy* for switched linear systems.

Let us consider a switched linear system[5] $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$. For a reminder, the trajectories $(\xi, \sigma) : \mathbb{T}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ of SwS with input $u : \mathbb{T}_{\geq 0} \to \mathbb{R}^n$ satisfy $\dot{\xi}(t) = A_{\sigma(t)}\xi(t) + B_{\sigma(t)}u(t)$ for all $t \in \mathbb{R}_{\geq 0}$ (if SwS is continuous-time) and $\xi(t+1) = A_{\sigma(t)}\xi(t) + B_{\sigma(t)}u(t)$ for all $t \in \mathbb{N}$ (if SwS is discrete-time), where $\xi : \mathbb{T}_{\geq 0} \to \mathbb{R}^n$ is the *continuous variable* and $\sigma : \mathbb{T}_{\geq 0} \to \Sigma$ is the *switching signal* of the trajectory, which specifies the *mode* $i \in \Sigma$ of SwS at each time $t \in \mathbb{T}_{\geq 0}$. For the sake of simplicity, we assume that all switched linear systems considered in this section are *under arbitrary switching*, which means that the set of admissible switching signals of SwS, denoted by $\mathcal{S}(\mathsf{SwS})$ (or $\mathcal{S}$ if SwS is clear from the context), is the set of right-continuous, piecewise constant functions from $\mathbb{T}_{\geq 0}$ to Σ (see also Subsection 1.3.1). Finally, given a switched linear system $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$, we let SwS° be the associated autonomous (or *open-loop*) switched linear system, defined by $\mathsf{SwS}^\circ \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ and $\mathcal{S}(\mathsf{SwS}^\circ) = \mathcal{S}(\mathsf{SwS})$.

Worst-case topological entropy of switched linear systems

In Subsection 3.2.1, we introduced the notion of topological entropy for a linear time-varying (LTV) system $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$, denoted by $h_{\text{top}}(\mathsf{Sys})$ (see Definition 3.1). We also saw, in Subsection 3.2.3, that a switched linear system $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ with a fixed switching signal $\sigma : \mathbb{T}_{\geq 0} \to \Sigma$ can be seen as a LTV system $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$ where $\hat{A} : \mathbb{T}_{\geq 0} \to \mathbb{R}^{n\times n}$ is defined by $\hat{A}(t) = A_{\sigma(t)}$ for all $t \in \mathbb{T}_{\geq 0}$, and we denoted its topological entropy by $h_{\text{top}}(\mathsf{SwS}; \sigma) = h_{\text{top}}(\mathsf{Sys})$. These notions allow us to define the worst-case topological entropy of a switched linear system.

Definition 3.13 (Worst-case topological entropy of a switched linear system)**.** *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *under arbitrary switching. The* worst-case topological entropy *of* SwS*, denoted by* $h_{\text{wc-top}}(\mathsf{Sys})$*, is defined as*

$$h_{\text{wc-top}}(\mathsf{SwS}) = \sup_{\sigma\in\mathcal{S}} h_{\text{top}}(\mathsf{SwS}; \sigma).$$

The example below illustrates the notion of worst-case topological entropy for switched linear systems.

Example 3.2. Consider the discrete-time switched linear system $\mathsf{SwS} \sim (\mathbb{R}^1, \{A_i\}_{i\in\Sigma})$ under arbitrary switching, with $\Sigma = \{1, 2\}$, and $A_1 = 1$ and $A_2 = 2$. Let

[5] We refer the reader to Section 1.3 for the notation and definitions related to switched systems.

$\sigma : \mathbb{N} \to \Sigma$ be the switching signal defined by $\sigma(t) = 1$ for all $t \in \mathbb{N}$ even, and $\sigma(t) = 2$ for all $t \in \mathbb{N}$ odd. The LTV system associated to SwS with switching signal σ corresponds to the one in Example 3.1, whose topological entropy was shown to be equal to 1/2. Hence, $h_{\text{top}}(\mathsf{SwS}; \sigma) = 1/2$. As for the worst-case topological entropy of SwS, it is not difficult to see that the switching signal $\sigma : \mathbb{N} \to \Sigma$ that maximizes the topological entropy is given by $\sigma(t) = 2$ for all $t \in \mathbb{N}$. In this case, $\chi(t, 0, x, \sigma) = 2^t x$ for all $t \in \mathbb{N}$ and $x \in \mathbb{R}^n$, where χ is the generator[6] of SwS. We deduce that $h(\mathsf{SwS}; \sigma) = \log_2 2 = 1$, so that $h_{\text{wc-top}}(\mathsf{SwS}) = 1$.

In Subsection 3.3.2, we give a closed-form expression for the worst-case topological entropy of a switched linear system, and we discuss the implications of this expression for the computation of the worst-case topological entropy of switched linear systems. Then, in Subsection 3.3.3, we explain the connections between the worst-case topological entropy and the minimal data rate for state estimation and stabilization of switched linear systems with a mode-dependent coder–decoder. These notions are reminded in the next subsubsection.

Minimal data rate for state estimation and stabilization of switched linear systems with a mode-dependent coder–decoder

The following definition of minimal data rate for state estimation of switched linear systems with a mode-dependent coder–decoder particularizes the definition of minimal data rate for state estimation of hybrid systems (see Definition 1.74 in Subsection 1.5.2) in the case of switched linear systems, and when the current mode of the system is known by the decoder via the universal output map.

Definition 3.14 (Minimal data rate for state estimation of a switched linear system with a mode-dependent coder–decoder)**.** *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *under arbitrary switching and a bounded set* $X_0 \subseteq \mathbb{R}^n$*. The* minimal data rate for state estimation *of* SwS *starting from* X_0 *with a* mode-dependent coder–decoder*, denoted by* $\mathcal{R}_{\text{est-md}}(\mathsf{SwS}, X_0)$*, is defined as the minimal data rate for state estimation of the hybrid system associated to* SwS*, with initial set* $X_0 \times \Sigma$*, and with respect to the cost function* $\mathfrak{C} : (\mathbb{R}^n \times \Sigma) \times (\mathbb{R}^n \times \Sigma) \to \mathbb{R}_{\geq 0}$ *defined by* $\mathfrak{C}(x_1, i_1, x_2, i_2) = \|x_1 - x_2\|$*, and via the universal output map* $\mathfrak{H} : \mathbb{R}^n \times \Sigma \to \Sigma$ *defined by* $\mathfrak{H}(x, i) = i$*.*

In other words, $\mathcal{R}_{\text{est-md}}(\mathsf{SwS}, X_0)$ in Definition 3.14 is defined as

$$\mathcal{R}_{\text{est-md}}(\mathsf{SwS}, X_0) = \sup_{\epsilon>0} \inf_{\mathsf{CoDec}} \mathcal{R}(\mathsf{CoDec}),$$

[6] For a reminder, see Definition 1.36 in Subsection 1.3.1.

where the infimum is over all mode-dependent coders–decoders CoDec that ϵ-observe SwS starting from X_0, and where $\mathcal{R}(\mathsf{CoDec})$ is the data rate of CoDec (see Definitions 1.71 and 1.73 in Subsection 1.5.1). Following the definitions in Subsection 1.5.1, a mode-dependent coder–decoder for the observation of SwS starting from X_0, with transmission period T_{t}, can be described as an ordered pair $((\Psi^{\mathrm{c}}_k)_{k\in\mathbb{N}}, (\Psi^{\mathrm{d}}_k)_{k\in\mathbb{N}})$ where for every $k \in \mathbb{N}$,

$$\Psi^{\mathrm{c}}_k : X_0 \times \Sigma^{[0,kT_{\mathrm{t}}]\,\cap\,\mathbb{T}} \to Y_{\mathrm{t}}$$

is the *coder function* at step k and

$$\Psi^{\mathrm{d}}_k : (Y_{\mathrm{t}})^{k+1} \times \bigcup\nolimits_{t\,\in\,[kT_{\mathrm{t}},(k+1)T_{\mathrm{t}})\,\cap\,\mathbb{T}} \Sigma^{[0,t]\,\cap\,\mathbb{T}} \to \mathbb{R}^n$$

is the *decoder function* at step k. At each time $t = kT_{\mathrm{t}}$, $k \in \mathbb{N}$, the coder outputs a symbol $e(k)$ defined by $e(k) = \Psi^{\mathrm{c}}_k(x, \sigma|_{[0,kT_{\mathrm{t}}]\,\cap\,\mathbb{T}})$ where x is the initial condition of the system and σ is the switching signal. The symbols are transmitted to the decoder, which produces at each time $t \in [kT_{\mathrm{t}}, (k+1)T_{\mathrm{t}}) \cap \mathbb{T}$, $k \in \mathbb{N}$, an estimate $\hat{\xi}(t) = \Psi^{\mathrm{d}}_k(e(0), \ldots, e(k), \sigma|_{[0,t]\,\cap\,\mathbb{T}})$ of the current state of the system. The coder–decoder $\mathsf{CoDec} = ((\Psi^{\mathrm{c}}_k)_{k\in\mathbb{N}}, (\Psi^{\mathrm{d}}_k)_{k\in\mathbb{N}})$ is said to ϵ-observe SwS starting from X_0 if for every trajectory $(\xi, \sigma) : \mathbb{T}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ of SwS with $\xi(0) \in X_0$ and every $t \in \mathbb{T}_{\geq 0}$, $\|\xi(t) - \hat{\xi}(t)\| \leq \epsilon$ where $\hat{\xi}(t)$ is defined as above, with $x = \xi(0)$. Finally, the data rate of CoDec is defined as the maximal number of bits per unit of time necessary to encode the symbols: $\mathcal{R}(\mathsf{CoDec}) = \frac{\lceil \log_2 |Y_{\mathrm{t}}| \rceil}{T_{\mathrm{t}}}$.

Similarly, we define the minimal data rate for stabilization of switched linear systems with a mode-dependent coder–decoder.

Definition 3.15 (Minimal data rate for stabilization of a switched linear system with a mode-dependent coder–decoder)**.** *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$ *under arbitrary switching. The* minimal data rate for stabilization *of* SwS *with a* mode-dependent coder–decoder*, denoted by* $\mathcal{R}_{\text{stab-md}}(\mathsf{SwS})$*, is defined as the minimal data rate for stabilization of the hybrid system associated to* SwS*, with respect to the cost function* $\mathfrak{C} : \mathbb{R}^n \times \Sigma \to \mathbb{R}_{\geq 0}$ *defined by* $\mathfrak{C}(x, i) = \|x\|$*, and via the universal output map* $\mathfrak{H} : \mathbb{R}^n \times \Sigma \to \Sigma$ *defined by* $\mathfrak{H}(x, i) = i$*.*

As for the case of state estimation, a mode-dependent coder–decoder for the stabilization of SwS, with transmission period T_{t}, can be described as an ordered pair $((\Psi^{\mathrm{c}}_k)_{k\in\mathbb{N}}, (\Psi^{\mathrm{d}}_k)_{k\in\mathbb{N}})$ where for every $k \in \mathbb{N}$,

$$\Psi^{\mathrm{c}}_k : \mathbb{R}^n \times \Sigma^{[0,kT_{\mathrm{t}}]\,\cap\,\mathbb{T}} \to Y_{\mathrm{t}}$$

is the *coder function* at step k and

$$\Psi^{\mathrm{d}}_k : (Y_{\mathrm{t}})^{k+1} \times \bigcup\nolimits_{t\,\in\,[kT_{\mathrm{t}},(k+1)T_{\mathrm{t}})\,\cap\,\mathbb{T}} \Sigma^{[0,t]\,\cap\,\mathbb{T}} \to \mathbb{R}^m$$

is the *decoder function* at step k. At each time $t = kT_{\mathrm{t}}$, $k \in \mathbb{N}$, the coder outputs a symbol $e(k)$ defined by $e(k) = \Psi^{\mathrm{c}}_k(x, \sigma|_{[0,kT_{\mathrm{t}}]\cap\mathbb{T}})$ where x is the initial condition of the system and σ is the switching signal. The symbols are transmitted to the decoder, which produces at each time $t \in [kT_{\mathrm{t}}, (k+1)T_{\mathrm{t}}) \cap \mathbb{T}$, $k \in \mathbb{N}$, a control input $u(t) = \Psi^{\mathrm{d}}_k(e(0), \ldots, e(k), \sigma|_{[0,t]\cap\mathbb{T}})$. The coder–decoder $\mathsf{CoDec} = ((\Psi^{\mathrm{c}}_k)_{k\in\mathbb{N}}, (\Psi^{\mathrm{d}}_k)_{k\in\mathbb{N}})$ is said to stabilize SwS if there is a class-$\mathcal{KL}$ function β such that for every $x \in \mathbb{R}^n$, $\sigma \in \mathcal{S}$ and $t \in \mathbb{T}_{\geq 0}$, $\|\chi(t, 0, x, \sigma, u)\| \leq \beta(\|x\|, t)$ where $u : \mathbb{T}_{\geq 0} \to \mathbb{R}^m$ is defined as above, and χ is the generator[7] of SwS.

3.3.2 Closed-form expression for the worst-case topological entropy of switched linear systems

For a continuous-time control-affine LTI system $\dot{\xi}(t) = A\xi(t) + Bu(t)$, with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, it is well known that the topological entropy of the open-loop system is given by

$$h_{\mathrm{top}}(A) = \log_2(e) \sum_{i=1}^{n} \max\{\mathrm{Re}(\lambda_i(A)), 0\} \qquad \text{(continuous-time)} \tag{3.4}$$

where $\lambda_1(A), \ldots, \lambda_n(A)$ are the eigenvalues of A, and it is also well known that $h_{\mathrm{top}}(A)$ coincides with the minimal data rate for state estimation of the open-loop system and with the minimal data rate for stabilization (see, e.g., Colonius, 2012, Section 4). Similar results hold for a discrete-time control-affine LTI system $\xi(t+1) = A\xi(t) + Bu(t)$: the topological entropy of the open-loop system is given by

$$h_{\mathrm{top}}(A) = \sum_{i=1}^{n} \max\{\log_2|\lambda_i(A)|, 0\} \qquad \text{(discrete-time)} \tag{3.5}$$

where $\lambda_1(A), \ldots, \lambda_n(A)$ are the eigenvalues of A, and $h_{\mathrm{top}}(A)$ coincides with the minimal data rate for state estimation of the open-loop system and with the minimal data rate for stabilization (see, e.g., Matveev and Savkin, 2009, Sections 2.4 and 2.5).

In this subsection, we present a closed-form expression, similar to (3.4)–(3.5), for the worst-case topological entropy of switched linear systems. The closed-form expression relies on the concept of joint spectral radius (see Definition 1.50 in Subsection 1.3.2) and *exterior powers of matrices*.

The subsection is organized as follows. First, we introduce the notion of exterior powers of matrices. Then, we present the closed-form expression for

[7] For a reminder, see Definition 1.36 in Subsection 1.3.1.

the worst-case topological entropy, and we discuss its consequences for the computation of the worst-case topological entropy of switched linear systems. Finally, we discuss the connections with related results in the literature at the end of this subsection.

Exterior algebras

The exterior algebra of a vector space V is an algebraic construction used to study the notions of area, volume, and their higher-dimensional analogues in V (see, e.g., Winitzki, 2010). In finite dimension, exterior algebras can be constructed from the exterior products of vectors in V, introduced below; since we restrict our attention to finite-dimensional spaces, and for simplicity of notation, we assume, without loss of generality, that $V = \mathbb{R}^n$.

Definition 3.16 (Exterior product of vectors)**.** *Let $v_1, \ldots, v_k \in \mathbb{R}^n$, with $k \in \mathbb{N}_{>0}$. The* exterior product *of $v_1, \ldots, v_k$, denoted by $v_1 \wedge \cdots \wedge v_k$, is the k-linear map from $(\mathbb{R}^n)^k$ to $\mathbb{R}$, defined by*

$$(v_1 \wedge \cdots \wedge v_k)(w_1, \ldots, w_k) = \det\left(\left[w_i^\top v_j\right]_{i,j=1}^{k,k}\right) \qquad \textit{for all } (w_1, \ldots, w_k) \in (\mathbb{R}^n)^k.$$

Definition 3.17 (k^{th} exterior power of a vector space)**.** *Consider a vector space $\mathbb{R}^n$ and let $k \in \{1, \ldots, n\}$. Let $\{v_1, \ldots, v_n\}$ be any basis of $\mathbb{R}^n$. The* k^{th} exterior power *of $\mathbb{R}^n$, denoted by $\Lambda^k\mathbb{R}^n$, is the vector space spanned by the exterior products $\{v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$. By convention, we also let $\Lambda^0\mathbb{R}^n = \mathbb{R}$.*

In particular, for all $k \in \{0, \ldots, n\}$, the dimension of $\Lambda^k\mathbb{R}^n$ is equal to $C(k, n) = n!/(k!(n-k)!)$. In numerical computations, it is convenient to treat $\Lambda^k\mathbb{R}^n$ as the coordinate space $\mathbb{R}^{C(k,n)}$. This can be done by fixing a basis $\mathcal{B}$ for $\Lambda^k\mathbb{R}^n$: e.g., $\mathcal{B} = \{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. If the elements of $\mathcal{B}$ are ordered with respect to the lexicographical order of their indices $(i_1, \ldots, i_k)$, then $\mathcal{B}$ is called the *canonical basis* of $\Lambda^k\mathbb{R}^n$.

Using the above, we define the concept of exterior power of a square matrix.

Definition 3.18 (k^{th} exterior power of a matrix)**.** *Let $A \in \mathbb{R}^{n\times n}$ and $k \in \{1, \ldots, n\}$. The k^{th}* exterior power *of A, denoted by $A^{\wedge k}$, is the unique linear map from $\Lambda^k\mathbb{R}^n$ to $\Lambda^k\mathbb{R}^n$ satisfying*

$$A^{\wedge k}(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k \qquad \textit{for all } (v_1, \ldots, v_k) \in (\mathbb{R}^n)^k.$$

By convention, we also let $A^{\wedge 0} = 1$.

Remark 3.1. The exterior power can be seen as a generalization of the concept of determinant: in particular, for all $A \in \mathbb{R}^{n\times n}$, it holds that $A^{\wedge n} = \det(A)$; see, e.g., Arnold (1998, Section 3.2.3).

For any $k \in \{0,\ldots,n\}$ and $A \in \mathbb{R}^{n\times n}$, using the canonical basis of $\Lambda^k\mathbb{R}^n$, $A^{\wedge k}$ can be represented by a $C(k,n)\times C(k,n)$ matrix. The *exterior power* of A, denoted by $A^\wedge$, is then defined as the $2^n\times 2^n$ matrix $A^\wedge = \operatorname{diag}\{A^{\wedge 0},\ldots,A^{\wedge n}\}$.

We also define the concept of reduced exterior power[8] of a square matrix.

Definition 3.19 (k^{th} reduced exterior power of a matrix)**.** *Let $A \in \mathbb{R}^{n\times n}$ and $k \in \{1,\ldots,n\}$. The k^{th}* reduced exterior power *of A, denoted by $A^{\odot k}$, is the unique linear map from $\Lambda^k\mathbb{R}^n$ to $\Lambda^k\mathbb{R}^n$ satisfying*

$$A^{\odot k}(v_1\wedge\cdots\wedge v_k) = \sum_{i=1}^{k} v_1\wedge\cdots\wedge v_{i-1}\wedge Av_i\wedge v_{i+1}\wedge\cdots\wedge v_k \qquad \text{for all } (v_1,\ldots,v_k)\in(\mathbb{R}^n)^k.$$

By convention, we also let $A^{\odot 0} = 0$.

Remark 3.2. The reduced exterior power can be seen as a generalization of the concept of trace: in particular, for all $A \in \mathbb{R}^{n\times n}$, it holds that $A^{\odot n} = \operatorname{trace}(A)$; see, e.g., Arnold (1998, Section 3.2.3).

For any $k \in \{0,\ldots,n\}$ and $A \in \mathbb{R}^{n\times n}$, using the canonical basis of $\Lambda^k\mathbb{R}^n$, $A^{\odot k}$ can be represented by a $C(k,n)\times C(k,n)$ matrix. The *reduced exterior power* of A, denoted by $A^\odot$, is then defined as the $2^n\times 2^n$ matrix $A^\odot = \operatorname{diag}\{A^{\odot 0},\ldots,A^{\odot n}\}$.

The following proposition, whose proof can be found in Arnold (1998), summarizes the properties of exterior algebras that will be needed in this work.

Proposition 3.20 (Arnold, 1998, Lemma 3.2.6)**.** *Let $k \in \{0,\ldots,n\}$ and $A,B \in \mathbb{R}^{n\times n}$.*

1. $I^{\wedge k} = I$, $(AB)^{\wedge k} = A^{\wedge k}B^{\wedge k}$, $(A^\top)^{\wedge k} = (A^{\wedge k})^\top$, $(A^\top)^{\odot k} = (A^{\odot k})^\top$.

2. *If A is upper-triangular/lower-triangular/diagonal/orthogonal, then so are $A^{\wedge k}$ and $A^{\odot k}$ (in the canonical basis of $\Lambda^k\mathbb{R}^n$).*

3. *The eigenvalues of $A^{\wedge k}$ are given by $\{\lambda_{i_1}(A)\lambda_{i_2}(A)\cdots\lambda_{i_k}(A) : 1 \le i_1 < i_2 < \ldots < i_k \le n\}$, where $\lambda_1(A),\ldots,\lambda_n(A)$ are the eigenvalues of A. In particular, $\rho(A^\wedge) = \prod_{i=1}^n \max\{|\lambda_i(A)|,1\}$ where $\rho(A^\wedge)$ is the spectral radius of $A^\wedge$. The eigenvalues of $A^{\odot k}$ are given by $\{\lambda_{i_1}(A)+\lambda_{i_2}(A)+\cdots+\lambda_{i_k}(A) : 1 \le i_1 < i_2 < \ldots < i_k \le n\}$.*

[8] This concept, which can be seen as the Lie derivative of the exterior product with respect to a linear vector field, seems to have no well-defined name in the literature. Therefore, for the purpose of this work, we chose the name of "reduced exterior power".

4. *The singular values of* $A^{\wedge k}$ *are given by* $\{\rho_{i_1}(A)\rho_{i_2}(A)\cdots\rho_{i_k}(A) : 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$, *where* $\rho_1(A),\ldots,\rho_n(A)$ *are the singular values of* A. *In particular,* $\|A^{\wedge}\| = \prod_{i=1}^{n}\max\{\rho_i(A),1\}$.

5. $e^{(A^{\odot k})} = (e^A)^{\wedge k}$. *Thus,* $e^{A^{\odot}} = (e^A)^{\wedge}$.

Finally, we define the exterior power of a switched linear system.

Definition 3.21 (Exterior power of a switched linear system)**.** *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$. *The* exterior power *of* SwS, *denoted by* $\mathsf{SwS}^{\wedge}$, *is defined by*

- $\mathsf{SwS}^{\wedge} \sim (\mathbb{R}^n, \{A_i^{\odot}\}_{i\in\Sigma})$ *and* $\mathcal{S}(\mathsf{SwS}^{\wedge}) = \mathcal{S}(\mathsf{SwS})$ *if* SwS *is continuous-time,*
- $\mathsf{SwS}^{\wedge} \sim (\mathbb{R}^n, \{A_i^{\wedge}\}_{i\in\Sigma})$ *and* $\mathcal{S}(\mathsf{SwS}^{\wedge}) = \mathcal{S}(\mathsf{SwS})$ *if* SwS *is discrete-time.*

Proposition 3.22. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$. *For every* $\sigma \in \mathcal{S}$ *and* $t_0, t_1 \in \mathbb{T}_{\geq 0}$, $t_1 \geq t_0$, *it holds that* $\mathring{\chi}(t_1, t_0, \sigma; \mathsf{SwS}^{\wedge}) = \mathring{\chi}(t_1, t_0, \sigma; \mathsf{SwS})^{\wedge}$, *where* $\mathring{\chi}(\cdot,\cdot,\cdot;\square)$ *is the fundamental matrix solution*[9] *of* $\square \in \{\mathsf{SwS}, \mathsf{SwS}^{\wedge}\}$.

Proof. Straightforward from Item 1 (and Item 5 for continuous-time systems) in Proposition 3.20 and from the definition of the exterior power of SwS (Definition 3.21). □

Main result and consequences

The main contribution of this subsection is the following theorem, which provides a closed-form expression for the worst-case topological entropy of switched linear systems.

Theorem 3.23. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *under arbitrary switching. It holds that*

$$h_{\text{wc-top}}(\mathsf{SwS}) = \log_2(e)\,\hat{\varrho}(\mathsf{SwS}^{\wedge})$$

where $\hat{\varrho}(\mathsf{SwS}^{\wedge})$ *is the joint spectral radius*[10] *of* $\mathsf{SwS}^{\wedge}$.

Proof. See Appendix A.3.5. □

[9] For a reminder, see Definition 1.47 in Subsection 1.3.

[10] For a reminder, see Definition 1.50 in Subsection 1.3.2.

We discuss below the implications of Theorem 3.23 for the numerical evaluation of the worst-case topological entropy of switched linear systems.

First of all, it follows from Theorem 3.23 that the worst-case topological entropy of a switched linear system depends continuously on its set of matrices. This property is important for numerical analysis as it ensures that small errors on the system model will not change too much the worst-case topological entropy of the system.

Corollary 3.24. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *under arbitrary switching. For any* $\epsilon > 0$*, there is* $\delta > 0$ *such that for any switched linear system* $\mathsf{SwS}' \sim (\mathbb{R}^n, \{A'_i\}_{i\in\Sigma})$ *under arbitrary switching, satisfying that* $\|A_i - A'_i\| \leq \delta$ *for all* $i \in \Sigma$*,* $|h_{\text{wc-top}}(\mathsf{SwS}) - h_{\text{wc-top}}(\mathsf{SwS}')| \leq \epsilon$*.*

Proof. The proof follows from the continuity of $A^\wedge$ or $A^\odot$ with respect to $A \in \mathbb{R}^{n\times n}$ and the continuity of the joint spectral radius with respect to bounded sets of matrices (straightforward consequence of Proposition 1.52 in Subsection 1.3.2; see also Jungers, 2009, Proposition 1.10). □

Secondly, Theorem 3.23 shows that the computation of the worst-case topological entropy can benefit from well-established algorithms for the computation of the joint spectral radius of switched linear systems.[11] Indeed, any of these algorithms can be used to approximate $\hat{\varrho}(\mathsf{SwS}^\wedge)$ from $\mathsf{SwS}^\wedge$. Furthermore, the computation of $\mathsf{SwS}^\wedge$ from SwS is straightforward from its definition (see Berger and Jungers, 2021c, Section 3.B). However, it should be noted that the dimension of $\mathsf{SwS}^\wedge$ increases exponentially with the dimension of SwS, and thus so will the complexity of estimating $\hat{\varrho}(\mathsf{SwS}^\wedge)$ (this is the curse of dimensionality!). In this regard, a simple and algorithm-independent way to speed up the estimation of $\rho(\mathsf{SwS}^\wedge)$, although not sufficient to fight the curse of dimensionality, is to observe that since the matrices $\{A_i^\wedge\}_{i\in\Sigma}$ and $\{A_i^\odot\}_{i\in\Sigma}$ are block diagonal, the computation of $\hat{\varrho}(\mathsf{SwS}^\wedge)$ can be decoupled among the different diagonal blocks (see, e.g., Jungers, 2009, Proposition 1.5).

Furthermore, there are cases for which the computation of the joint spectral radius is straightforward; for instance, when $\{A_i\}_{i\in\Sigma}$ is a set of normal/upper-triangular/lower-triangular matrices. By combining these results with the properties of the exterior power of matrices (see Proposition 3.20), we get

[11] A wide range of methods, of very different natures, have been proposed in the last decades to evaluate the joint spectral radius of a set of matrices; see, e.g., Jungers (2009, Section 2.3), Sun and Ge (2011, Section 2.4) and Vankeerberghen et al. (2014). While theoretical discouraging results exist for the computation of the joint spectral radius in general (see, e.g., Jungers, 2009, Section 2.2), these methods turn out to be extremely powerful in practice and to provide high-accuracy approximation algorithms for the joint spectral radius.

efficient ways to compute the worst-case topological entropy of normal/upper-triangular/lower-triangular switched linear systems.

Corollary 3.25. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *under arbitrary switching. Assume that for each* $i \in \Sigma$, A_i *is a normal matrix, and let* $\lambda_1(A_i), \ldots, \lambda_n(A_i)$ *be the eigenvalues of* A_i. *Then, it holds that*

$$h_{\text{wc-top}}(\mathsf{SwS}) = \max_{i\in\Sigma} \sum\nolimits_{j=1}^{n} \max\{\log_2(e)\,\mathrm{Re}(\lambda_j(A_i)), 0\} \qquad \text{(continuous-time case)}$$

$$h_{\text{wc-top}}(\mathsf{SwS}) = \max_{i\in\Sigma} \sum\nolimits_{j=1}^{n} \max\{\log_2|\lambda_j(A_i)|, 0\} \qquad \text{(discrete-time case).}$$

The same holds for switched linear systems with upper-triangular/lower-triangular matrices.

Proof. See Appendix A.3.6. □

Similarly, the theory of p-dominance for switched linear systems (introduced in Section 2.2) can be helpful to reduce the complexity of computing the worst-case topological entropy of p-dominant switched linear systems.

Corollary 3.26. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *under arbitrary switching. Assume that* SwS *is* p*-dominant (Definition 2.8 in Subsection 2.2.2) with automaton* $\mathsf{Aut} = (Q, \Sigma, \Theta)$ *and set of rates* $\{\gamma_\theta\}_{\theta\in\Theta}$*, and assume that* Aut *is cycle-stable with respect to* $\{\gamma_\theta\}_{\theta\in\Theta}$ *(Definition 2.14 in Subsection 2.2.2). Then, it holds that*

$$h_{\text{top}}(\mathsf{SwS}) = \max_{k\in\{0,\ldots,p\}} \log_2(e)\,\hat{\varrho}(\mathsf{SwS}^{\wedge k})$$

where $\mathsf{SwS}^{\wedge k}$ *is defined as* $\mathsf{SwS}^{\wedge}$ *but with* $\{A_i^{\wedge k}\}_{i\in\Sigma}$ $(\{A_i^{\odot k}\}_{i\in\Sigma})$ *instead of* $\{A_i^{\wedge}\}_{i\in\Sigma}$ $(\{A_i^{\odot}\}_{i\in\Sigma})$.

Proof. Straightforward from the characterization of the asymptotic behavior of switched linear systems that are p-dominant with a cycle-stable automaton (see Theorem 2.15 in Subsection 2.2.2) and from Item 4 in Proposition 3.20. □

Numerical examples illustrating the computation of the worst-case topological entropy of switched linear systems, using Theorem 3.23 and Corollary 3.25, are presented in Subsection 3.3.4.

Remark 3.3. In view of the closed-form expression of Theorem 3.23, one might legitimately think that a similar formula holds for the topological entropy of a switched linear system $\mathsf{SwS} = (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ with a fixed switching signal $\sigma : \mathbb{T}_{\geq 0} \to \Sigma$, namely that $h_{\text{top}}(\mathsf{SwS};\sigma) = \limsup_{T\to\infty} \frac{1}{T} \log_2 \|\mathring{\chi}(T,0,\sigma)^{\wedge}\|$. We present a counter-example in Appendix A.3.7 showing that this is not the case in general.

Related works

The worst-case topological entropy of a switched linear system provides an upper bound on the topological entropy of the system with any fixed switching signal. The question of estimating the topological entropy of switched linear systems with a fixed switching signal is studied thoroughly in Yang et al. (2020). Because it is assumed that the switching signal is fixed, the bounds on the topological entropy obtained in Yang et al. (2020) are in general better than the worst-case topological entropy. However, in some "ill-conditioned" cases (e.g., triangular systems with large differences among the diagonal entries), the bounds available in Yang et al. (2020) can be more conservative than the worst-case topological entropy (which can be computed efficiently, e.g., for triangular systems; see Corollary 3.25).

Exterior algebras have also received attention in systems and control theory; namely, in the study of the Lyapunov exponents (see, e.g., Arnold, 1998, and Barreira, 2017) and entropy-related properties of dynamical systems (see, e.g., Kozlovski, 1998, and Kawan, 2013). For instance, we note the formula by Kozlovski for the topological entropy of a discrete-time autonomous dynamical system $\mathsf{Sys} = (\mathbb{R}^n, f)$, with C^∞ map $f : \mathbb{R}^n \to \mathbb{R}^n$, over a compact forward invariant set $X \subseteq \mathbb{R}^n$:

$$h_{\mathrm{top}}(\mathsf{Sys}, X) = \lim_{T\to\infty} \frac{1}{T} \log_2 \int_X \left\| \tfrac{\partial f^T}{\partial x}(x)^\wedge \right\| \mathrm{d}x.$$

Theorem 3.23 shows, among others things, that the integral can be replaced by a maximum over all switching signals in the case of the worst-case topological entropy of switched linear systems, and enables practical computation using the stability theory of switched linear systems.

3.3.3 Minimal data rate for state estimation and stabilization of switched linear systems with a mode-dependent coder–decoder

In this subsection, we show that the minimal data rate for state estimation of an autonomous switched linear system with a mode-dependent coder–decoder coincides with its worst-case topological entropy. Similarly, the minimal data rate for stabilization of a feedback stabilizable switched linear system with a mode-dependent coder–decoder coincides with the worst-case topological entropy of the open-loop switched linear system. Moreover, we describe the implementation of a *practical* (i.e., implementable) coder–decoder that observes, or stabilizes, the system and whose data rate can be arbitrarily close to the optimal data rate (equal to the worst-case topological entropy). These results are encapsulated in the following two theorems.

Theorem 3.27. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *under arbitrary switching, and a bounded set* $X_0 \subseteq \mathbb{R}^n$ *with nonempty interior. It holds that* $\mathcal{R}_{\text{est-md}}(\mathsf{SwS}, X_0) = h_{\text{wc-top}}(\mathsf{SwS})$*.*
Moreover, for any $R > h_{\text{wc-top}}(\mathsf{SwS})$ *and* $\epsilon > 0$*, there is a* practical *mode-dependent coder–decoder with data rate smaller than or equal to R, that* ϵ*-observes* SwS*.*

Theorem 3.28. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$ *under arbitrary switching. Assume that* SwS *is feedback stabilizable with a static controller*[12] *(see Definitions 1.17 and 1.19 in Subsection 1.1.3). Then, it holds that* $\mathcal{R}_{\text{stab-md}}(\mathsf{SwS}) = h_{\text{wc-top}}(\mathsf{SwS}^\circ)$*.*
Moreover, for any $R > h_{\text{wc-top}}(\mathsf{SwS})$*, there is a* practical *mode-dependent coder–decoder* CoDec *with data rate smaller than or equal to R, that stabilizes* SwS *with exponential asymptotic rate of convergence, meaning that there is* $\mu > 0$ *and a class-*$\mathcal{K}$ *function g such that for any* $x \in \mathbb{R}^n$*,* $\sigma \in \mathcal{S}$ *and* $t \in \mathbb{T}_{\geq 0}$*,*

$$\|\chi(t, 0, x, \sigma; \mathsf{SwS}\|\mathsf{CoDec})\| \leq g(\|x\|)e^{-\mu t}, \tag{3.6}$$

where $\mathsf{SwS}\|\mathsf{CoDec}$ *is the closed-loop system obtained from the feedback composition of* SwS *and* CoDec *(see Definition 1.70 in Subsection 1.5.1).*

We present the proof of Theorem 3.28 only, as the proof of Theorem 3.27 is identical. The proof that $\mathcal{R}_{\text{stab-md}}(\mathsf{SwS}) \geq h_{\text{wc-top}}(\mathsf{SwS}^\circ)$ in Theorem 3.28 is presented in Appendix A.3.8. The proof that $\mathcal{R}_{\text{stab-md}}(\mathsf{SwS}) \leq h_{\text{wc-top}}(\mathsf{SwS}^\circ)$ in Theorem 3.28 will follow from the next subsubsection, where we describe a practical coder–decoder satisfying the assertions of the theorem.

Practical coder–decoder

Consider a switched linear system $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$ under arbitrary switching, and assume that SwS is feedback stabilizable with a static controller, denoted by $\kappa : \mathbb{R}^n \times \Sigma \to \mathbb{R}^m$. Let $R > h_{\text{wc-top}}(\mathsf{SwS}^\circ)$. We describe the implementation of a practical coder–decoder satisfying (3.6), and whose data rate is smaller than or equal to R.

In the following, we let $\mathbb{B}$ be the centered unit Euclidean ball. First, we describe for any matrix $A \in \mathbb{R}^{n\times n}$ and resolution $\alpha > 0$, a finite-points quantizer for the reachable set of A from $\mathbb{B}$; see the algorithm in Figure 3.5.

[12] In applications, static feedback stabilizing controllers for switched linear systems are generally linear, meaning that for each $i \in \Sigma$, there is $F_i \in \mathbb{R}^{m\times n}$ such that the switched linear system $\mathsf{SwS}' \sim (\mathbb{R}^n, \{A'_i\}_{i\in\Sigma})$, defined by $A'_i = A_i + B_i F_i$ for each $i \in \Sigma$ and $\mathcal{S}(\mathsf{SwS}') = \mathcal{S}(\mathsf{SwS})$, is stable; but this is not required for this theorem.

Definition 3.29 (Quantizer associated to a matrix)**.** *For any $A \in \mathbb{R}^{n\times n}$ and $\alpha > 0$, the quantizer implemented by the algorithm in Figure 3.5 is called the* quantizer with resolution α associated to A *and is denoted by $Q_{\alpha,A} : \mathbb{R}^n \to \Xi_{\alpha,A}$. The set $\Xi_{\alpha,A}$ is called the* range *of $Q_{\alpha,A}$, and we denote its cardinality by $\hat{m}(\alpha, A) = |\Xi_{\alpha,A}|$.*

The following result follows directly from the definition of $Q_{\alpha,A}$ in Definition 3.29.

Lemma 3.30. *For every $A \in \mathbb{R}^{n\times n}$ and $\alpha > 0$, the quantizer $Q_{\alpha,A} : \mathbb{R}^n \to \Xi_{\alpha,A}$ satisfies that*

- *for all $x \in \mathbb{B}$, $\|Ax - Q_{\alpha,A}(Ax)\| \leq \alpha$, and $Q_{\alpha,A}(x) = 0$ if $\|x\| \leq \frac{\alpha}{n^{1/2}}$;*
- *$\hat{m}(\alpha, A) = \prod_{i=1}^n \left(2\left[\!\left[\frac{n^{1/2}}{2\alpha}\rho_i(A)\right]\!\right] + 1\right)$ where $\rho_1(A), \ldots, \rho_n(A)$ are the singular values of A.*

Proof. Straightforward from the definition of $Q_{\alpha,A}$. □

By combining the second item of the above lemma with the closed-form expression for the worst-case topological entropy, we obtain the following bound on the cardinality of the quantizer associated to the fundamental matrix solution of a switched linear system.

Lemma 3.31. *Consider a switched linear system $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ under arbitrary switching, and a resolution $\alpha > 0$. Let $R > h_{\text{wc-top}}(\mathsf{SwS})$. There is $T \in \mathbb{T}_{\geq 0}$ such that for every $\sigma \in \mathcal{S}$ and $T' \in \mathbb{T}$, $T' \geq T$, it holds that $\hat{m}(\alpha, \mathring{\chi}(T', 0, \sigma)) \leq 2^{\lfloor RT' \rfloor}$.*

Proof. See Appendix A.3.9. □

Now, let us define a coder–decoder satisfying (3.6), and whose data rate is smaller than or equal to R. First, we define some parameters, depending on the system SwS and its stabilizing controller κ. Then, we describe the implementation of the coder and the decoder. Finally, we discuss the correctness of the implementation.

Parameters: Fix $\alpha \in (0, 1)$ and let $T_1 \in \mathbb{T}_{\geq 0}$ be such that for every $x \in \mathbb{B}$, $\sigma \in \mathcal{S}$ and $T' \in \mathbb{T}$, $T' \geq T_1$, it holds that $\|\chi(T', 0, x, \sigma; \mathsf{SwS}\|\kappa)\| \leq \alpha$, where $\mathsf{SwS}\|\kappa$ is the closed-loop system obtained from the composition with the static controller κ. Fix $R > h_{\text{wc-top}}(\mathsf{SwS}^\circ)$ and let $T_2 \in \mathbb{T}_{\geq 0}$ be such that for every $\sigma \in \mathcal{S}$ and $T' \in \mathbb{T}$, $T' \geq T_2$, it holds that $\hat{m}(\alpha, \mathring{\chi}(T', 0, \sigma)) \leq 2^{\lfloor RT' \rfloor}$ (Lemma 3.31). Finally, let $T = \max\{T_1, T_2\}$ be the period of the coder–decoder.

Remark 3.4. We assume here, for the sake of simplicity of presentation, that the quantity T_2 above is computed a priori, but this quantity can in fact be computed *on the fly*; see Berger and Jungers (2021c, Section 3.D).

Coder and decoder implementations: For the parameters defined above, the associated coder and decoder are implemented by the algorithms in Figure 3.7 (the reader may find useful to refer to Figure 3.6, where the different quantities involved in the algorithms are represented). For the sake of simplicity of presentation, we assume that the trajectories of the system start in $\mathbb{B}$, as the general case can be handled by adding a "capturing phase" (via a "zooming-out" procedure) at the beginning of the stabilization process, as explained in Liberzon (2014, Section 4.3). Also, we assume that the coder–decoder is without transmission delay. Again, this assumption is made for simplicity of presentation, but is not necessary; see Berger and Jungers (2021c, Section 3.D).

Correctness of the coder–decoder: The proof that the coder–decoder described in Figure 3.7 has a data rate smaller than or equal to R and that it satisfies (3.6) is presented in Appendix A.3.10.

3.3.4 Numerical experiments

In this subsection, we illustrate the application of the results of Subsections 3.3.2 and 3.3.3 on several numerical examples.

Worst-case topological entropy

We use the results of Subsection 3.3.2 to compute the worst-case topological entropy of switched linear systems with general and triangular matrices.

Example 3.3. Consider the continuous-time switched linear system $\mathsf{SwS} \sim (\mathbb{R}^2, \{A_i\}_{i\in\Sigma})$ under arbitrary switching, with $\Sigma = \{1, 2\}$, and $A_1 = \left[\begin{smallmatrix} 0.1 & 2.0 \\ 0.5 & 0.1 \end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix} -0.5 & 0.5 \\ 2.0 & 0.0 \end{smallmatrix}\right]$. The exterior power of SwS is given by $\mathsf{SwS}^\wedge \sim (\mathbb{R}^4, \{A_i^\odot\}_{i\in\Sigma})$ where

$$A_1^\odot = \begin{bmatrix} 1 & & & \\ & 0.1 & 2.0 & \\ & 0.5 & 0.1 & \\ & & & 0.2 \end{bmatrix} \quad \text{and} \quad A_2^\odot = \begin{bmatrix} 1 & & & \\ & -0.5 & 0.5 & \\ & 2.0 & 0.0 & \\ & & & -0.5 \end{bmatrix}.$$

We have used the *JSR Toolbox* (see Vankeerberghen et al., 2014), combined with Protasov and Jungers (2013, Theorem 3), to estimate the joint spectral radius of $\mathsf{SwS}^\wedge$. This provided the interval $\hat{\varrho}(\mathsf{SwS}^\wedge) \in (1.21, 1.22)$. Hence, by Theorem 3.23, it follows that $h_{\text{wc-top}}(\mathsf{SwS}) \in \log_2(e)(1.21, 1.22) \approx (1.75, 1.76)$ bits per unit of time.

Example 3.4. Consider the discrete-time switched linear system $\mathsf{SwS} \sim (\mathbb{R}^2, \{A_i\}_{i\in\Sigma})$ under arbitrary switching, with $\Sigma = \{1,2\}$, and $A_1 = \begin{bmatrix} 3 & 1 \\ & 0.1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1.1 & 1 \\ & 2 \end{bmatrix}$. Since A_1 and A_2 are upper-triangular, we may apply Corollary 3.25. We deduce that the worst-case topological entropy of the SwS is equal to $\log_2 3 = 1.5850$. The reader will check that the same result can be obtained by applying directly Theorem 3.23; indeed the exterior powers of SwS is given by $\mathsf{SwS}^\wedge \sim (\mathbb{R}^4, \{A_i^\wedge\}_{i\in\Sigma})$ where

$$A_1^\wedge = \begin{bmatrix} 1 & & & \\ & 3 & 1 & \\ & & 0.1 & \\ & & & 0.3 \end{bmatrix} \quad \text{and} \quad A_2^\wedge = \begin{bmatrix} 1 & & & \\ & 1.1 & 1 & \\ & & 2 & \\ & & & 2.2 \end{bmatrix},$$

and the joint spectral radius of a discrete-time upper-triangular switched linear system is given by the largest absolute value of the diagonal entries of its matrices (see, e.g., Jungers, 2009, Proposition 2.3).

Stabilization with a mode-dependent coder–decoder

In this subsubsection, we illustrate the use of the coder–decoder described in Figure 3.7 for the stabilization of a continuous-time switched linear system.

Example 3.5. Consider the continuous-time switched linear system $\mathsf{SwS} \sim (\mathbb{R}^2, \mathbb{R}^1, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$ under arbitrary switching, with $\Sigma = \{1,2\}$, and $A_1 = \begin{bmatrix} 0.1 & 2.0 \\ 0.5 & 0.1 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.5 & 0.5 \\ 2.0 & 0.0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This system is stabilizable with the linear static controller $\kappa : \mathbb{R}^2 \times \Sigma \to \mathbb{R}^1$ defined by $\kappa(x,i) = K_i x$ where $K_1 = \begin{bmatrix} -1.261 & -1.261 \end{bmatrix}$ and $K_2 = \begin{bmatrix} -2.5 & -0.823 \end{bmatrix}$.

Note that the open-loop system SwS° is the switched linear system studied in Example 3.3, which was shown to satisfy $h_{\text{wc-top}}(\mathsf{SwS}^\circ) \in \log_2(e)(1.21, 1.22)$ bits per unit of time (see Example 3.3). Hence, by Theorem 3.28, it follows that for any $R \geq 1.22 \log_2(e)$, the coder–decoder described in Figure 3.7, with data rate R, stabilizes the system. A sample execution of SwS controlled by the coder–decoder with data rate $R = 3\log_2(e)$ is represented in Figure 3.8-a. A comparison of the rate of convergence for different values of the data rate of the coder–decoder is presented in Figure 3.8-b. As intuitively expected, we observe that the norm of ξ decreases more rapidly when the data rate is higher.

Stabilization with a coder–decoder and controlled switching signal

In this subsubsection, we illustrate the application of the mode-dependent setting for the stabilization of a continuous-time switched linear system with controlled switching signal and when the observation of the state is subject to data-rate constraints.

Example 3.6. Consider the continuous-time switched linear system $\mathsf{SwS} \sim (\mathbb{R}^2, \{A_i\}_{i\in\Sigma})$ with $\Sigma = \{1, 2\}$, and $A_1 = \begin{bmatrix} 0.1 & -1.0 \\ 1.0 & 0.1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$ (inspired from Jungers and Mason, 2017, Example 2.3). Assume that we want to stabilize this system by controlling the switching signal; that is, at each time instant, the controller can choose the mode of the system, and the objective is to drive the state of the system to zero. The effect of each matrix of SwS is represented in Figure 3.9-a,b. A switching control strategy to stabilize the system is the following (see also Figure 3.9-c for an illustration):

- **Rotation mode:** While the angle between $\xi(t)$ and the horizontal axis is not small enough (say $> \pi/8$), apply the matrix A_1 ("rotation + divergence") until the angle becomes small enough ($\leq \pi/8$). When this is the case, the controller switches to a "convergence mode";

- **Convergence mode:** Apply the matrix A_2 ("horizontal convergence + vertical divergence") until the angle between $\xi(t)$ and the horizontal axis becomes too large (say $> \pi/6$). During this phase, since $\xi(t)$ is sufficiently horizontal, the state gets closer to the origin, but at the same time it becomes less horizontal. Once the angle between $\xi(t)$ and the horizontal axis is too large ($> \pi/6$), the controller switches back the the "rotation mode".

Our goal is to design a coder–decoder that stabilizes the system using quantized measurements of the state and using the switching signal as control input. For that, we use the mode-dependent coder–decoder described in Figure 3.7 for the estimation of the state. Note that, for this application, the assumption that the current mode is known by the decoder is automatically satisfied since the decoder chooses the current mode of the system.

In terms of data rate requirement: since one can hardly predict in advance what will be the sequence of modes of the controlled system, it is natural to consider the worst-case scenario to deduce the data rate that will allows us to estimate the state of the system with exponentially decreasing error. Therefore, we compute the worst-case topological entropy of $\mathsf{SwS}^\circ \sim (\mathbb{R}^2, \{A_i\}_{i\in\Sigma})$ under arbitrary switching. Using the formula of Theorem 3.23, we find that $h_{\mathrm{top}}(\mathsf{SwS}^\circ) = \log_2(e)$ (one can show that $\hat{\varrho}(\mathsf{SwS}^\circ) = 1$, by using the Lyapunov function $V(x) = \|x\|$, satisfying $\dot{V}(\xi(t)) \leq V(\xi(t))$). Thus, for any $R > \log_2(e)$, there is a mode–controlling coder–decoder, with data rate R, that stabilizes the system. A sample execution of SwS controlled by the coder–decoder with data rate $R = 2.5\log_2(e)$ is represented in Figure 3.10-a. A comparison of the rate of convergence for different values of the data rate of the coder–decoder is presented in Figure 3.10-b. As intuitively expected, we observe that the norm of

ξ decreases more rapidly when the data rate is higher.

3.4 Quantized control of switched linear systems with mode-oblivious coder–decoder

In this section, we study the problem of state estimation and stabilization of switched linear systems when only limited information about the state and the mode of the system is available. We restrict our attention to continuous-time switched linear systems, since the case of discrete-time systems is much easier to handle. First, we show that switched linear systems, with or without constraints on the switching signal, have in general an infinite topological entropy, implying that they are in general not observable with arbitrary accuracy with a finite data rate. Then, we show that switched linear systems under arbitrary switching, i.e., with no constraint on the switching signal, are in general not stabilizable with a finite data rate. Drawing on this result, we restrict our attention to switched linear systems satisfying a fairly mild slow-switching assumption, namely that the switching signal has an average dwell time bounded away from zero. We show that under this assumption, switched linear systems that are stabilizable in the classical sense remain stabilizable with a finite data rate.

The section is organized as follows. In Subsection 3.4.1, we remind the notions of topological entropy and minimal data rate for state estimation and stabilization of switched linear systems. In Subsection 3.4.2, we present the negative results regarding the state estimation of switched linear systems and the stabilization of switched linear systems under arbitrary switching with a mode-oblivious coder–decoder. In Subsection 3.4.3, we show that switched linear systems with average dwell time bounded away from zero preserve their stabilizability properties under data-rate constraints. In particular, we describe a practical coder–decoder that stabilizes the system when the system is stabilizable in the absence of data-rate constraints and the average dwell time is bounded away from zero. Finally, in Subsection 3.4.4, we demonstrate the applicability of our results on numerical examples.

Notation. In this section, all considered switched linear systems are continuous-time switched linear systems, and thus for the sake of brievety, we will refer to them simply as *switched linear systems.* The restriction of a function $f : A \to B$ to a set $A' \subseteq A$ is denoted by $f|_{A'}$. $\lceil\cdot\rceil$, $\lfloor\cdot\rfloor$ and $[\![\cdot]\!]$ are the ceil, floor and round functions.

3.4.1 Problem setting

We introduce the problem of interest of this section: namely, the quantized observation and stabilization of switched linear systems when the switching signal of the system is not directly observed by the decoder (mode-oblivious coder–decoder). We also discuss the notion of topological entropy (different from the worst-case topological entropy introduced in the previous section) for switched linear systems.

Let us consider a switched linear system[13] $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$. For a reminder, the trajectories $(\xi, \sigma) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ of SwS with input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ satisfy $\dot{\xi}(t) = A_{\sigma(t)}\xi(t) + B_{\sigma(t)}u(t)$ for all $t \in \mathbb{R}_{\geq 0}$, where $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is the *continuous variable* and $\sigma : \mathbb{R}_{\geq 0} \to \Sigma$ is the *switching signal* of the trajectory, which specifies the *mode* $i \in \Sigma$ of SwS at each time $t \in \mathbb{R}_{\geq 0}$. The set of admissible complete switching signals of SwS is denoted by $\mathcal{S}(\mathsf{SwS})$, or $\mathcal{S}$ if SwS is clear from the context (see Definition 1.34 in Subsection 1.3.1). We remind that all switching signals of SwS are right-continuous, piecewise constant functions from $\mathbb{R}_{\geq 0}$ to Σ. Given a right-continuous, piecewise constant function $\sigma : \mathbb{R}_{\geq 0} \to \Sigma$, and $t_0, t_1 \in \mathbb{R}_{>0}$, $t_1 \geq t_0$, we let $N_\sigma(t_1, t_0)$ be the number of discontinuity points of σ in $[t_0, t_1)$. Given $\tau_{\mathrm{a}} > 0$, a switched linear system SwS is said to have *average dwell time* τ_{a} if all its switching signals have average dwell time τ_{a} with some parameter $N_\circ \geq 0$, meaning that for all $\sigma \in \mathcal{S}$, and $t_1, t_0 \in \mathbb{R}_{>0}$, $t_1 \geq t_0$, it holds that $N_\sigma(t_1, t_0) \leq N_\circ + \frac{t_1 - t_0}{\tau_{\mathrm{a}}}$ (see Definition 1.40 in Subsection 1.3.1). Finally, given a switched linear system $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$, we let SwS° be the associated autonomous (or *open-loop*) switched linear system, defined by $\mathsf{SwS}^\circ \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ and $\mathcal{S}(\mathsf{SwS}^\circ) = \mathcal{S}(\mathsf{SwS})$.

Topological entropy of switched linear systems

The following definition of topological entropy particularizes the one for hybrid systems (see Definition 1.78 in Subsection 1.5.2) to switched linear systems.

Definition 3.32 (Topological entropy of a switched linear system)**.** *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *and a bounded set* $X_0 \subseteq \mathbb{R}^n$*. The* topological entropy *of* SwS *starting from* X_0*, denoted by* $h_{\mathrm{top}}(\mathsf{SwS}, X_0)$*, is defined as the topological entropy of the hybrid system associated to* SwS*, with initial set* $X_0 \times \Sigma$*, and with respect to the cost function* $\mathfrak{C} : (\mathbb{R}^n \times \Sigma) \times (\mathbb{R}^n \times \Sigma) \to \mathbb{R}_{\geq 0}$ *defined by* $\mathfrak{C}(x_1, i_1, x_2, i_2) = \|x_1 - x_2\|$*.*

[13] We refer the reader to Section 1.3 for the notation and definitions related to switched systems.

In other words, $h_{\text{top}}(\mathsf{SwS}, X_0)$ in Definition 3.32 is defined as

$$h_{\text{top}}(\mathsf{SwS}, X_0) = \sup_{\epsilon>0} \limsup_{T\to\infty} \frac{1}{T} \log_2 s_{\text{span}}(\epsilon, T; X_0), \tag{3.7}$$

where $s_{\text{span}}(\epsilon, T; X_0)$ is the smallest cardinality of an (ϵ, T)-spanning set for SwS starting from X_0, that is, the minimal number of functions from $[0, T)$ to $\mathbb{R}^n$ necessary to approximate, with accuracy ϵ on the interval $[0, T)$, the "ξ" component of all trajectories $(\xi, \sigma) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ of SwS with $\xi(0) \in X_0$ (see Definition 1.76 in Subsection 1.5.2 for details). Equivalently, $h_{\text{top}}(\mathsf{SwS}, X_0)$ can be defined as

$$h_{\text{top}}(\mathsf{SwS}, X_0) = \sup_{\epsilon>0} \limsup_{T\to\infty} \frac{1}{T} \log_2 s_{\text{sep}}(\epsilon, T; X_0), \tag{3.8}$$

where $s_{\text{sep}}(\epsilon, T; X_0)$ is the largest cardinality of an (ϵ, T)-separated set for SwS starting from X_0, that is, the maximal number of trajectories $(\xi, \sigma) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ of SwS satisfying $\xi(0) \in X_0$ and whose "ξ" components are ϵ-distinguishable on $[0, T)$ (see Definition 1.77 and Proposition 1.79 in Subsection 1.5.2 for details).

Remark 3.5. Consider a switched linear system $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ and a bounded set $X_0 \subseteq \mathbb{R}^n$ with nonempty interior. By contrast to the case of LTV systems (see Subsection 3.2.1), it seems an open question whether $h_{\text{top}}(\mathsf{SwS}, X_0)$ depends on X_0 or not. It nevertheless holds that $h_{\text{top}}(\mathsf{SwS}, X_0)$ is invariant by positive scaling of the initial set X_0, meaning that for any $c > 0$, $h_{\text{top}}(\mathsf{SwS}, cX_0) = h_{\text{top}}(\mathsf{SwS}, X_0)$. It follows that $h_{\text{top}}(\mathsf{SwS}, X_0)$ is maximal (over all bounded $X_0 \subseteq \mathbb{R}^n$) when X_0 contains the origin in its interior. For the sake of brievety, the proof is omitted.

In Subsection 3.4.2, we will see that the topological entropy of switched linear systems is infinite whenever the system is unstable. We will also see that a similar result holds for the minimal data rate for state estimation of switched linear systems, but not for the minimal data rate for stabilization of switched linear systems, under mild assumptions on the systems. These two notions of minimal data rate for state estimation and stabilization are reminded in the subsubsection below.

Minimal data rate for state estimation and stabilization of switched linear systems

The following definition of minimal data rate for state estimation of switched linear systems (with a mode-oblivious coder–decoder) particularizes the definition of minimal data rate for state estimation of hybrid systems (see Definition

1.74 in Subsection 1.5.2) in the case of switched linear systems, and when the universal output map is the empty map.

Definition 3.33 (Minimal data rate for state estimation of a switched linear system). *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *and a bounded set* $X_0 \subseteq \mathbb{R}^n$. *The* minimal data rate for state estimation *of* SwS *starting from* X_0, *denoted by* $\mathcal{R}_{\mathrm{est}}(\mathsf{SwS}, X_0)$, *is defined as the minimal data rate for state estimation of the hybrid system associated to* SwS, *with initial set* $X_0 \times \Sigma$, *and with respect to the cost function* $\mathfrak{C} : (\mathbb{R}^n \times \Sigma) \times (\mathbb{R}^n \times \Sigma) \to \mathbb{R}_{\geq 0}$ *defined by* $\mathfrak{C}(x_1, i_1, x_2, i_2) = \|x_1 - x_2\|$.

In other words, $\mathcal{R}_{\mathrm{est}}(\mathsf{SwS}, X_0)$ in Definition 3.33 is defined as

$$\mathcal{R}_{\mathrm{est}}(\mathsf{SwS}, X_0) = \sup_{\epsilon>0} \inf_{\mathsf{CoDec}} \mathcal{R}(\mathsf{CoDec}),$$

where the infimum is over all coders–decoders CoDec that ϵ-observe SwS starting from X_0, and where $\mathcal{R}(\mathsf{CoDec})$ is the data rate of CoDec (see Definitions 1.71 and 1.73 in Subsection 1.5.1). Following the definitions in Subsection 1.5.1, a (mode-oblivious) coder–decoder for the observation of SwS starting from X_0, with transmission period T_{t}, can be described as an ordered pair $((\Psi^{\mathrm{c}}_k)_{k\in\mathbb{N}}, (\Psi^{\mathrm{d}}_k)_{k\in\mathbb{N}})$ where for every $k \in \mathbb{N}$,

$$\Psi^{\mathrm{c}}_k : X_0 \times \Sigma^{[0,kT_{\mathrm{t}}]} \to Y_{\mathrm{t}}$$

is the *coder function* at step k and

$$\Psi^{\mathrm{d}}_k : (Y_{\mathrm{t}})^{k+1} \times [kT_{\mathrm{t}}, (k+1)T_{\mathrm{t}}) \to \mathbb{R}^n$$

is the *decoder function* at step k. At each time $t = kT_{\mathrm{t}}$, $k \in \mathbb{N}$, the coder outputs a symbol $e(k)$ defined by $e(k) = \Psi^{\mathrm{c}}_k(x, \sigma|_{[0,kT_{\mathrm{t}}]})$ where x is the initial condition of the system and σ is the switching signal. The symbols are transmitted to the decoder, which produces at each time $t \in [kT_{\mathrm{t}}, (k+1)T_{\mathrm{t}})$, $k \in \mathbb{N}$, an estimate $\hat{\xi}(t) = \Psi^{\mathrm{d}}_k(e(0), \ldots, e(k), t)$ of the current state of the system. The coder–decoder $\mathsf{CoDec} = ((\Psi^{\mathrm{c}}_k)_{k\in\mathbb{N}}, (\Psi^{\mathrm{d}}_k)_{k\in\mathbb{N}})$ is said to ϵ-observe SwS starting from X_0 if for every trajectory $(\xi, \sigma) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ of SwS with $\xi(0) \in X_0$ and every $t \in \mathbb{R}_{\geq 0}$, $\|\xi(t) - \hat{\xi}(t)\| \leq \epsilon$ where $\hat{\xi}(t)$ is defined as above, with $x = \xi(0)$. Finally, the data rate of CoDec is defined as the maximal number of bits per unit of time necessary to encode the symbols: $\mathcal{R}(\mathsf{CoDec}) = \frac{\lceil \log_2 |Y_{\mathrm{t}}| \rceil}{T_{\mathrm{t}}}$.

Similarly, we define the minimal data rate for stabilization of switched linear systems (with a mode-oblivious coder–decoder).

Definition 3.34 (Minimal data rate for stabilization of a switched linear system). *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$.

The minimal data rate for stabilization *of* SwS*, denoted by* $\mathcal{R}_{\mathrm{stab}}(\mathsf{SwS})$*, is defined as the minimal data rate for stabilization of the hybrid system associated to* SwS*, with respect to the cost function* $\mathfrak{C} : \mathbb{R}^n \times \Sigma \to \mathbb{R}_{\geq 0}$ *defined by* $\mathfrak{C}(x, i) = \|x\|$*.*

As for the case of state estimation, a (mode-oblivious) coder–decoder for the stabilization of SwS, with transmission period T_t, can be described as an ordered pair $((\Psi_k^\mathrm{c})_{k\in\mathbb{N}}, (\Psi_k^\mathrm{d})_{k\in\mathbb{N}})$ where for every $k \in \mathbb{N}$,

$$\Psi_k^\mathrm{c} : \mathbb{R}^n \times \Sigma^{[0,kT_\mathrm{t}]} \to Y_\mathrm{t}$$

is the *coder function* at step k and

$$\Psi_k^\mathrm{d} : (Y_\mathrm{t})^{k+1} \times [kT_\mathrm{t}, (k+1)T_\mathrm{t}) \to \mathbb{R}^m$$

is the *decoder function* at step k. At each time $t = kT_\mathrm{t}$, $k \in \mathbb{N}$, the coder outputs a symbol $e(k)$ defined by $e(k) = \Psi_k^\mathrm{c}(x, \sigma|_{[0,kT_\mathrm{t}]})$ where x is the initial condition of the system and σ is the switching signal. The symbols are transmitted to the decoder, which produces at each time $t \in [kT_\mathrm{t}, (k+1)T_\mathrm{t})$, $k \in \mathbb{N}$, a control input $u(t) = \Psi_k^\mathrm{d}(e(0), \ldots, e(k), t)$. The coder–decoder $\mathsf{CoDec} = ((\Psi_k^\mathrm{c})_{k\in\mathbb{N}}, (\Psi_k^\mathrm{d})_{k\in\mathbb{N}})$ is said to stabilize SwS if there is a class-$\mathcal{KL}$ function β such that for every $x \in \mathbb{R}^n$, $\sigma \in \mathcal{S}$ and $t \in \mathbb{R}_{\geq 0}$, $\|\chi(t, 0, x, \sigma, u)\| \leq \beta(\|x\|, t)$ where $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is defined as above, and χ is the generator[14] of SwS.

3.4.2 Obstacles to state estimation and stabilization of switched linear systems with a mode-oblivious coder–decoder

We start with some negative results regarding the topological entropy and the minimal data rate for state estimation of switched linear systems.

Theorem 3.35. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$*, a bounded set* $X_0 \subseteq \mathbb{R}^n$ *with nonempty interior, and a dwell time* $\tau_\mathrm{a} > 0$*. Assume that* $\mathcal{S}$ *contains all switching signals from* $\mathbb{R}_{\geq 0}$ *to* Σ *with absolute dwell time* τ_a *(see Definition 1.40 in Subsection 1.3.1). Also, assume that* $\{A_i\}_{i\in\Sigma}$ *contains at least one unstable matrix, and that for every* $x \in \mathbb{R}^n \setminus \{0\}$*, there are* $i_1, i_2 \in \Sigma$ *such that* $A_{i_1}x \neq A_{i_2}x$*. Then, it holds that* $h_{\mathrm{top}}(\mathsf{SwS}, X_0) = \infty$*.*

Proof. See Appendix A.3.11. □

Remark 3.6. Let us mention that a result similar to the one in Theorem 3.35 has been proved in the recent preprint by Sibai and Mitra (2020, Theorem 5).

[14] For a reminder, see Definition 1.36 in Subsection 1.3.1.

Corollary 3.36. *With the hypothesis of Theorem 3.35, it holds that* $\mathcal{R}_{\mathrm{est}}(\mathsf{SwS}, X_0) = \infty$.

Proof. This follows from $\mathcal{R}_{\mathrm{est}}(\mathsf{SwS}, X_0) \geq h_{\mathrm{top}}(\mathsf{SwS}, X_0)$ (see Proposition 1.80 in Subsection 1.5.2). □

The important feature of switched linear systems responsible for the above negative results is their *non-determinism*, which implies that from any state, there is in general an infinite set of states that can be reached by the system over a finite horizon. In the case of switched linear systems, the diameter of the set of states that can be reached from a given state depends on the norm of the state in question. The assumption that there is at least one unstable matrix involved in the system implies that this diameter stays bounded away from zero, at least along some trajectories. In terms of quantized observation, this implies that, along such trajectories, the uncertainty on the current state of the system between two transmission times cannot be arbitrarily close to zero, even if the data rate goes to infinity, resulting in the infeasibility of observing the system with a finite data rate.

Now, we discuss the question of the minimal data rate for stabilization of switched linear systems. The situation is quite different from the one of state estimation. Indeed, in this case, we can control the norm of the trajectories (this is the goal of stabilization), so that the uncertainty on the state of the system between the transmission times is not the main issue. However, we need to take into account the uncertainty on the mode of the system, which implies that even if the state of the system is known accurately, one does not know which control input to apply to stabilize the system (since the effect of the control input will depend on the mode of the system). The example below illustrates the fact that the uncertainty on the current mode of the system can prevent the system to be stabilized with a finite data rate.

Example 3.7. Consider the continuous-time switched linear system $\mathsf{SwS} \sim (\mathbb{R}^1, \mathbb{R}^1, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$ under arbitrary switching, with $\Sigma = \{1, 2\}$, and $A_1 = A_2 = 0$, $B_1 = -1$ and $B_2 = 1$. In other words, the trajectories of SwS satisfy $\dot{\xi}(t) = -u(t)$ if $\sigma(t) = 1$ and $\dot{\xi}(t) = u(t)$ if $\sigma(t) = 2$. This system is somehow the most basic switched control-affine system, and is clearly stabilizable, e.g., with the linear static controller $\kappa : \mathbb{R}^1 \times \Sigma \to \mathbb{R}^1$ defined by $\kappa(x, i) = -B_i x$.

Proposition 3.37. *Let* SwS *be as in Example 3.7. It holds that* $\mathcal{R}_{\mathrm{stab}}(\mathsf{SwS}) = \infty$.

Proof. Let CoDec be a coder–decoder for SwS. Let $T \in \mathbb{R}_{>0}$ and let $\mathcal{U}_T$ be the set of all distinct input functions that can be produced by CoDec during the interval $[0,T)$. By definition of a coder–decoder, $\mathcal{U}_T$ is finite.

Now, for each $\nu > 0$, let $\sigma_\nu : \mathbb{R}_{\geq 0} \to \Sigma$ be the switching signal that oscillates between mode 1 and mode 2 with frequency $2/\nu$: that is, for every $t \in \mathbb{R}_{\geq 0}$, $\sigma_\nu(t) = 1$ if $2k\nu \leq t < (2k+1)\nu$ for some $k \in \mathbb{N}$, and $\sigma_\nu(t) = 2$ if $(2k+1)\nu \leq t < 2(k+1)\nu$ for some $k \in \mathbb{N}$. Then, by using an adaptation of the proof of the Riemann–Lebesgue lemma (see, e.g., Teschl, 2021, Corollary 14.5), one can show that for any integrable function $u : [0,T) \to \mathbb{R}$, it holds that $\int_0^T B_{\sigma_\nu(t)} u(t)\,\mathrm{d}t \to 0$ when $\nu \to 0$. Since $\mathcal{U}_T$ is finite, this implies that for any $\epsilon > 0$, there is $\nu > 0$ such that $|\int_0^T B_{\sigma_\nu(t)} u(t)\,\mathrm{d}t| < \epsilon$ for all $u \in \mathcal{U}_T$. Thus, for every $x \in \mathbb{R}^1$ and $u \in \mathcal{U}_T$, it holds that $|\chi(T,0,x,\sigma_\nu,u;\mathsf{SwS})| > |x| - \epsilon$. Since T and ϵ were arbitrary, this implies that CoDec does not stabilize SwS. Since CoDec was arbitrary, this shows that $\mathcal{R}_{\mathrm{est}}(\mathsf{SwS}) = \infty$, concluding the proof. □

One way to limit the uncertainty on the mode of the system is to impose slow-switching conditions on the switching signal of the system. In particular, in the next subsection, we show that any feedback stabilizable switched linear system with average dwell time bounded away from zero can be stabilized with a finite data rate.

3.4.3 Stabilization of switched linear systems with dwell time with a mode-oblivious coder–decoder

As shown in Example 3.7, switched linear systems under arbitrary switching are in general not stabilizable with a finite data rate. For this reason, we consider switched linear systems with a positive average dwell time. We make the following assumption on the system, which accounts for the fact the the system with average dwell time τ_{a} is stabilizable with a static feedback controller.

Assumption 3.38. *Given a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$ *and a dwell time* $\tau_{\mathrm{a}} > 0$*, we assume that there is a static feedback controller* $\kappa : \mathbb{R}^n \times \Sigma \to \mathbb{R}^m$*, and constants* $D \geq 0$ *and* $\mu_1, \mu_2 > 0$ *such that* $\mu_1/\tau_{\mathrm{a}} < \mu_2$ *and for every* $x \in \mathbb{R}^n$*,* $\sigma : \mathbb{R}_{\geq 0} \to \Sigma$ *and* $t \in \mathbb{R}_{\geq 0}$*, it holds that*

$$\|\chi(t,0,x,\sigma;\mathsf{SwS}\|\kappa)\| \leq D\|x\| e^{\mu_1 N_\sigma(t,0) - \mu_2 t} \tag{3.9}$$

where $\mathsf{SwS}\|\kappa$ *is the closed-loop system (assumed to be under arbitrary switching*[15]*) obtained from the composition with the static controller* κ*.*

[15] This assumption is made only for the well definition of $\chi(\cdot,\cdot,\cdot,\sigma;\mathsf{SwS}\|\kappa)$ for all $\sigma : \mathbb{R}_{\geq 0} \to \Sigma$.

With SwS and κ as in the assumption, (3.9) implies that the closed-loop system $\mathsf{SwS}\|\kappa$ is exponentially stable for all switching signal $\sigma : \mathbb{R}_{\geq 0} \to \Sigma$ with average dwell time τ_{a}.

Remark 3.7. A controller satisfying Assumption 3.38 can be derived for instance if SwS admits a *multiple control Lyapunov function*; see, e.g., Liberzon (2003) or Lin and Antsaklis (2009). An interesting situation, is when SwS admits a *common* control Lyapunov function; in this case, (3.9) is satisfiable with $\mu_1 = 0$, so that Assumption 3.38 holds for any $\tau_{\mathrm{a}} > 0$.

We are now able to present the main result of this subsection, which states that any switched linear system with positive average dwell time satisfying Assumption 3.38 is stabilizable with a finite data rate.

Theorem 3.39. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$ *with average dwell time* $\tau_{\mathrm{a}} > 0$*, and let Assumption 3.38 hold. There is a (mode-oblivious) coder–decoder* CoDec *that stabilizes* SwS *with exponential asymptotic rate of convergence, meaning that there is* $\mu > 0$ *and a class-*$\mathcal{K}$ *function* g *such that for any* $x \in \mathbb{R}^n$*,* $\sigma \in \mathcal{S}$ *and* $t \in \mathbb{T}_{\geq 0}$*,*

$$\|\chi(t, 0, x, \sigma; \mathsf{SwS}\|\mathsf{CoDec})\| \leq g(\|x\|)e^{-\mu t}, \tag{3.10}$$

where $\mathsf{SwS}\|\mathsf{CoDec}$ *is the closed-loop system obtained from the feedback composition of* SwS *and* CoDec *(see Definition 1.70 in Subsection 1.5.1). In particular, it holds that* $\mathcal{R}_{\mathrm{stab}}(\mathsf{SwS}) < \infty$*.*

We will provide a constructive proof of Theorem 3.39. More precisely, in the subsubsection below, we describe the implementation of a coder–decoder satisfying the assertions of the theorem. A precise upper bound on the minimal data rate for stabilization of the system will be derived in due course of the description of the coder–decoder; see (3.12). As for the decay rate μ in (3.10), it will be obtained in the proof of the correctness of the proposed coder–decoder; see (A.25) in Appendix A.3.12. The gain function g, however, will not be explicitly defined but its existence will be demonstrated. As a class-$\mathcal{K}$ function, g satisfies $g(r) \to 0$ when $r \to 0$. However, as it will be clear from the proof of its existence, $g(r)$ is not Lipschitz continuous at $r = 0$. This lack of regularity is not due to a potential sub-optimality of the proposed coder–decoder, or to the switching nature of the system, but is intrinsic to *any* finite-data-rate stabilization scheme for linear systems (including LTI systems); see for instance Colonius (2012, Proposition 2.2).

Description of a coder–decoder satisfying Theorem 3.39

Consider a switched linear system $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$ with average dwell time $\tau_{\mathrm{a}} > 0$ Let Assumption 3.38 hold with controller $\kappa : \mathbb{R}^m \times \Sigma \to \mathbb{R}^m$ and $D \geq 0$ and $\mu_1, \mu_2 > 0$. We describe the implementation of a coder–decoder CoDec that satisfies (3.10) for some $\mu > 0$ and some class-$\mathcal{K}$ function g. First, we define some parameters, which depend on SwS, τ_{a}, κ, D, μ_1 and μ_2. Then, we describe the implementation of the coder and the controller. Finally, we discuss the correctness of the implementation.

Parameters: Let $\nu = \frac{1}{2}\max_{i\in\Sigma} \lambda_{\max}(A_i + A_i^\top)$, and let $\Delta_A = \max_{i_1,i_2\in\Sigma} \|A_{i_1} - A_{i_2}\|$ and $\Delta_B = \max_{i_1,i_2\in\Sigma} \|B_{i_1} - B_{i_2}\|$. Also, define $L = \sup\{\, \|\kappa(x,i)\|/\|x\| : i \in \Sigma,\ x \in \mathbb{R}^n \setminus \{0\}\,\}$. Pick $T \in \mathbb{R}_{>0}$, $\alpha > 0$ and $p \in \mathbb{N}_{>0}$ such that

$$\boxed{De^{-\mu_2 pT} + e^{\nu pT}\alpha + (1 + \tfrac{1}{\alpha})\varepsilon(p,T) < \theta e^{-\mu_1 pT/\tau_{\mathrm{a}}}} \tag{3.11}$$

for some $\theta \in (0,1)$, where $\varepsilon(p,T) = e^{\nu pT} T \frac{pT}{\tau_{\mathrm{a}}} D(\Delta_A + \Delta_B L)$.[16] Finally, let Q_α be the quantizer $Q_{\alpha,I}$ in Definition 3.29 where I is the $n \times n$ identity matrix, and let Ξ_α be the set of quantizing points of Q_α, whose cardinality satisfies $|\Xi_\alpha| \leq \hat{m}_\alpha \doteq \big(2\big[\!\big[\frac{n^{1/2}}{2\alpha}\big]\!\big] + 1\big)^n$ (see Lemma 3.30).

We will build a coder–decoder CoDec with transmission period T, that stabilizes the system and operates at data rate

$$\mathcal{R}(\mathsf{CoDec}) = \tfrac{1}{T}\left(\left\lceil \tfrac{1}{p}\lceil \log_2((p+1)\hat{m}_\alpha)\rceil\right\rceil + \lceil \log_2(|\Sigma|)\rceil\right). \tag{3.12}$$

Coder and decoder implementations: For the parameters defined above, the associated coder and decoder are implemented by the algorithms in Figure 3.11. For the sake of simplicity of presentation, we assume that the trajectories of the system start in the centered unit Euclidean ball, as the general case can be handled by adding a "capturing phase" (via a "zooming-out" procedure) at the beginning of the stabilization process, as explained in Liberzon (2014, Section 4.3). Also, we assume that the coder–decoder is without transmission delay. Again, this assumption is made for simplicity of presentation, but can be alleviated at the cost of increasing slightly the data rate of the coder–decoder.

The implementation deserves the following explanations. If $\sigma : \mathbb{R}_{\geq 0} \to \Sigma$ is the switching signal of the system, then $\hat{\sigma} : \mathbb{R}_{\geq 0} \to \Sigma$ is the sample-and-hold switching signal built for the observations of σ by coder at the transmission times, that is, for all $t \in \mathbb{R}_{\geq 0}$, it holds that $\hat{\sigma}(t) = \sigma(\lfloor t/T \rfloor T)$. Also, for $j_0, j_1 \in \mathbb{N}$, $j_1 \geq j_0$, we let $N_\sigma^{\mathrm{sw}}(T; j_0, j_1)$ be the number of transmission intervals

[16] A strategy for choosing T, α, p is: first, choose $T' = pT$ large enough so that $De^{(\mu_1/\tau_{\mathrm{a}} - \mu_2)T'} < 1$. Then, for this T', choose α, p such that $e^{\nu T'}\alpha$ and $\varepsilon(p, T'/p)$ are small enough for (3.11) to be satisfied with $\theta < 1$.

during which σ switches at least once, that is,

$$N_\sigma^{\mathrm{sw}}(T; j_0, j_1) = |\{j \in [j_0, j_1) \cap \mathbb{N} : N_\sigma((j+1)T, jT) > 0\}|. \tag{3.13}$$

By definition, it holds that $N_\sigma^{\mathrm{sw}}(T, j_1, j_0) \leq j_1 - j_0$. Finally, for all $v \in \{0, \ldots, p\}$, we define

$$\bar{a}(v) = De^{\mu_1 v - \mu_2 pT}, \quad \text{and} \quad \bar{b}(v) = e^{\nu pT} vTe^{\mu_1 v} D(\Delta_A + \Delta_B L). \tag{3.14}$$

Correctness of the coder–decoder: The proof that the coder–decoder described in Figure 3.7 has data rate as in (3.12) and that it satisfies (3.6) is presented in Appendix A.3.12.

3.4.4 Numerical experiments

In this subsubsection, we illustrate the use of the coder–decoder described in Figure 3.11 for the stabilization of a continuous-time switched linear system.

Example 3.8. Consider the continuous-time switched linear system $\mathsf{SwS} \sim (\mathbb{R}^2, \mathbb{R}^1, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$, with $\Sigma = \{1, 2\}$, and $A_1 = \begin{bmatrix} 0.1 & -1.0 \\ 1.5 & 0.1 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.5 & 2.0 \\ -1.5 & 0.0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This system satisfies Assumption 3.38 with $D = 1$, $\mu_1 = 0$, $\mu_2 = 0.15$ and with the linear static controller $\kappa : \mathbb{R}^2 \times \Sigma \to \mathbb{R}^1$ defined by $\kappa(x, i) = K_i x$ where $K_1 = \begin{bmatrix} -0.43 & -0.43 \end{bmatrix}$ and $K_2 = \begin{bmatrix} -0.38 & -0.52 \end{bmatrix}$.

First, we have simulated the system with average dwell time $\tau_{\mathrm{a}} = 1.0$ s. We have used the values $T = 0.008$, $\alpha = 0.05$ and $p = 100$ for the parameters of the coder–decoder, which satisfy (3.11). With these parameters, the data rate of the coder–decoder is 145 bits/s. A sample execution of SwS with this average dwell time, controlled by the coder–decoder, is represented in Figure 3.12-a. We observe that the state of the system converges to zero, as predicted.

Then, we have simulated the system with a smaller average dwell time, namely $\tau_{\mathrm{a}} = 0.25$ s. We have used the values $\tau_s = 0.002$, $\alpha = 0.05$ and $n = 400$ for the parameters of the coder–decoder, which satisfy (3.11). The data rate of the coder–decoder is 523 bits/s. A sample execution of SwS with this average dwell time, controlled by the coder–decoder, is represented in Figure 3.12-b. Again, we observe that the sampled trajectory converges to zero, as predicted.

3.5 Conclusions

In this chapter, we studied the interaction of switching and quantization in control problems. Namely, we studied the question of quantized control for switched linear systems, and considered two different settings for this control problem, accounting for different assumptions on the information about the operating mode of the system available to the decoder. The results obtained for the different settings are summarized in Table 3.1.

The mode-dependent setting is the one for which we obtained the strongest positive results. In particular, we related the minimal data rate for state estimation and stabilization with a entropy-based notion, namely the worst-case topological entropy. We also showed that these quantities could be approximated numerically with arbitrary accuracy. On the other hand, we demonstrated several negative results for the quantized control of switched linear systems in the mode-oblivious setting. This motivated the introduction of additional assumptions on the switching mechanism of the system to make the problem of quantized stabilization with mode-oblivious coder–decoder tractable. Similar assumptions were already considered in previous works on this topic, but not necessarily motivated by counter-examples. Under these assumptions, we provided sufficient data rate bounds for stabilization, and we described the implementation of a practical coder–decoder achieving stabilization whenever the system is stabilizable in the absence of data-rate constraints. Since the mode-dependent setting is the one for which we have the most information about the system, it is not surprising that the strongest positive results were obtained for this setting, while the strongest negative results were obtained for the setting with the weakest information structure.

An interesting direction for further research is to investigate further applications of the mode-dependent setting for the control of switched linear systems. We think for instance to applications in the context of controlled or constrained switching (see namely Example 3.6 for a proof-of-concept). Indeed, the worst-case topological entropy provides an upper bound on the minimal data rate for stabilization of these systems; it would be worth investigating whether it is possible to improve this bound or define more efficient coders–decoders for the stabilization of these systems if the switching signal can be controlled or constrained in some way by the decoder.

It would also be interesting to investigate improvements of the coder–decoder described in the mode-oblivious setting. For instance, one could consider the use of Lyapunov functions (as in Liberzon, 2014, Wakaiki and Yamamoto, 2014, and Yang and Liberzon, 2018, for instance), refine the analysis of the propagation of reachable sets during transmission intervals by using tools from multilinear algebra (as in the mode-dependent case), or consider additional assumptions on the system.

	Mode-dependent		Mode-oblivious
	Fixed switching signal (LTV systems)	Free switching signal	Free switching signal
Entropy	Classical definition of topological entropy (Definition 3.1)	Worst-case topological entropy = maximal topological entropy over all switching signals (Definition 3.13) Always finite + "computable" close-form expression (Theorem 3.23)	Classical definition of topological entropy for non-deterministic systems (Definition 3.32) Can be infinite (Theorem 3.35)
State estimation	Minimal data rate = topological entropy (Theorem 3.3) Implementation may require unbounded memory	Minimal data rate = worst-case topological entropy + can be reached by a practical coder–decoder (Theorem 3.27)	May be not observable with finite data rate, even under slow-switching assumptions (Corollary 3.36)
Stabilization	Minimal data rate = topological entropy (straightforward extension of Theorem 3.3) Implementation may require unbounded memory	Minimal data rate = worst-case topological entropy of open-loop system + can be reached by a practical coder–decoder (Theorem 3.28)	May be not stabilizable with finite data, under arbitrary switching (Theorem 3.7) Stabilizable with finite data + description of a practical coder–decoder, under nonzero average dwell-time assumption (Theorem 3.39)

Table 3.1: Summary of the results regarding the quantized control of switched linear systems.

Coder

Input: A LTV system $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$, a bounded set $X_0 \subseteq \mathbb{R}^n$. An accuracy $\epsilon > 0$, a "data rate" $\alpha > h_{\text{top}}(\mathsf{Sys})$ and a period $T \in \mathbb{R}_{>0}$.

Output: Symbols $e(k)$ to be sent to the decoder at each time $t = kT$, $k \in \mathbb{N}$.

Algorithm:

Let $x = \xi(0)$ be the initial condition of ξ.

Let $\hat{x}_{-1} = 0 \in \mathbb{R}^n$ and $\tilde{E}_{-1} = X_0$.

Loop: at time $t = kT$ **for** $k = 0, 1, 2, \ldots$

Let $T_k \in \mathbb{R} \cup \{\infty\}$ be the largest time such that $s_{\text{cov}}(\epsilon, T_k; X_0) \leq 2^{\alpha(k+1)T}$.

Let E_k be a minimal (ϵ, T_k)-cover of $\tilde{E}_{k-1}$.

Let $\hat{y}_k \in E_k$ be such that $x - \hat{x}_{k-1} \in \mathbb{B}_{\mathsf{Sys},T_k}(\hat{y}_k, \epsilon)$.

Encode $\hat{y}_k$ as a symbol $e(k)$ and send it to the decoder.

Let $\hat{x}_k = \hat{x}_{k-1} + \hat{y}_k$.

Let $\tilde{E}_k = \mathbb{B}_{\mathsf{Sys},T_k}(\hat{x}_k, \epsilon)$.

Decoder

Input: A LTV system $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$, a bounded set $X_0 \subseteq \mathbb{R}^n$. An accuracy $\epsilon > 0$, a "data rate" $\alpha > h_{\text{top}}(\mathsf{Sys})$ and a period $T \in \mathbb{R}_{>0}$.

Output: A estimate $\hat{\xi}(t)$ at each time $t \in \mathbb{R}_{\geq 0}$ satisfying $\|\xi(t) - \hat{\xi}(t)\| \leq \epsilon$.

Algorithm:

Let $\hat{x}_{-1} = 0 \in \mathbb{R}^n$ and $\tilde{E}_{-1} = X_0$.

Loop: at time $t = kT$ **for** $k = 0, 1, 2, \ldots$

Let $T_k \in \mathbb{R} \cup \{\infty\}$ be the largest time such that $s_{\text{cov}}(\epsilon, T_k; X_0) \leq 2^{\alpha(k+1)T}$.

Let $\hat{y}_k$ be decoded from the symbol $e(k)$ sent by the coder.[†]

Let $\hat{x}_k = \hat{x}_{k-1} + \hat{y}_k$.

For each $t \in [kT, (k+1)T)$, let $\hat{\xi}(t) = \chi(t, x_k)$.

Let $\tilde{E}_k = \mathbb{B}_{\mathsf{Sys},T_k}(\hat{x}_k, \epsilon)$.

Figure 3.4: Implementation of the coder–decoder. $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is the trajectory of the system and starts in X_0. [†]For each $k \in \mathbb{N}$, the decoder can compute E_k, and thus it can compute $\hat{y}_k$ from $e(k)$.

Input: A matrix $A \in \mathbb{R}^{n\times n}$ and a resolution $\alpha > 0$.
Output: A finite-points quantizer $Q : \mathbb{R}^n \to \Xi \subseteq \mathbb{R}^n$ with resolution α on $A\mathbb{B}$.
Algorithm:
Let $\bar{\alpha} = 2\alpha/n^{1/2}$.
Let $USV^\top$ be a SVD of A, with $U, V \in \mathbb{R}^{n\times n}$ orthogonal and $S = \mathrm{diag}(\rho_1, \ldots, \rho_n) \in \mathbb{R}^{n\times n}$ diagonal.
For each $j \in \{1, \ldots, n\}$, let $S_j = \{-[\![\rho_j/\bar{\alpha}]\!], \ldots, [\![\rho_j/\bar{\alpha}]\!]\}$.
Let $\Xi = \bar{\alpha}U(S_1 \times \cdots \times S_n)$. Let $Q : \mathbb{R}^n \to \Xi \subseteq \mathbb{R}^n$ be defined by $Q(x) = \arg\min_{\hat{x}\in\Xi} \|x - \hat{x}\|$.

Figure 3.5: Quantizer associated to a square matrix.

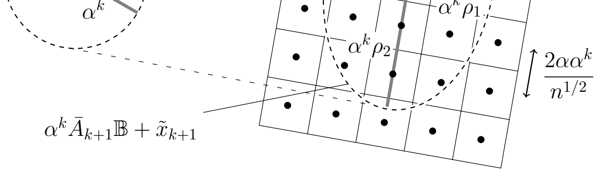


Figure 3.6: The different quantities involved in the implementation of the coder–decoder described in Figure 3.7. The black points represent the quantized points, i.e., the set $\Xi_{\alpha,\bar{A}_k}$ scaled by α_k and shifted by $\tilde{x}_{k+1}$.

Coder

Input: A switched linear system $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$ under arbitrary switching, a static controller $\kappa : \mathbb{R}^n \times \Sigma \to \mathbb{R}^m$ for SwS. A resolution $\alpha > 0$ and a period $T \in \mathbb{T}_{>0}$.

Output: Symbols $e(k)$ to be sent to the decoder at each time $t = kT$, $k \in \mathbb{N}$.

Algorithm:

Let $\hat{x}_0 = 0 \in \mathbb{R}^n$ and $n_0 = \|\hat{x}_0\| = 0$. Send "empty symbol" and wait until $t = T$.

Loop: at time $t = kT$ **for** $k = 1, 2, \ldots$

- Let $\tilde{x}_k = n_{k-1}\chi(kT, (k-1)T, \hat{x}_{k-1}/n_{k-1}, \sigma; \mathsf{SwS}\|\kappa)$ (or $\tilde{x}_k = 0$ if $n_{k-1} = 0$).†
- Measure $\xi(kT)$ and let $y_k = \xi(kT) - \tilde{x}_k$.
- Let $\bar{A}_k = \mathring{\chi}(kT, (k-1)T, \sigma; \mathsf{SwS}^\circ)$.†
- Let $\hat{y}_k = Q_{\alpha,\bar{A}_k}(y_k/\alpha^{k-1})$.
- Encode $\hat{y}_k$ as a symbol $e(k)$ and send it to the decoder.
- Let $\hat{x}_k = \tilde{x}_k + \alpha^{k-1}\hat{y}_k$.

Decoder

Input: A switched linear system $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$ under arbitrary switching, a static controller $\kappa : \mathbb{R}^n \times \Sigma \to \mathbb{R}^m$ for SwS. A resolution $\alpha > 0$ and a period $T \in \mathbb{T}_{>0}$.

Output: A control input $u(t)$ at each time $t \in \mathbb{T}_{\geq 0}$ that is applied to the system.

Algorithm:

Let $\hat{x}_0 = 0 \in \mathbb{R}^n$ and $n_0 = \|\hat{x}_0\| = 0$. Apply the input $u \equiv 0 \in \mathbb{R}^m$ until $t = T$.

Loop: at time $t = kT$ **for** $k = 1, 2, \ldots$

- Let $\tilde{x}_k = n_{k-1}\chi(kT, (k-1)T, \hat{x}_{k-1}/n_{k-1}, \sigma; \mathsf{SwS}\|\kappa)$ (or $\tilde{x}_k = 0$ if $n_{k-1} = 0$).†
- Let $\bar{A}_k = \mathring{\chi}(kT, (k-1)T, \sigma; \mathsf{SwS}^\circ)$.†
- Let $\hat{y}_k$ be decoded from the symbol $e(k)$ sent by the coder.‡
- Let $\hat{x}_k = \tilde{x}_k + \alpha^{k-1}\hat{y}_k$ and $n_k = \|\hat{x}_k\|$.
- For each $t \in [kT, (k+1)T) \cap \mathbb{T}$, let $u(t) = n_k\,\kappa(\chi(t, kT, \hat{x}_k/n_k, \sigma; \mathsf{SwS}\|\kappa), \sigma(t))$.†

Figure 3.7: Implementation of the coder–decoder. $(\xi, \sigma) : \mathbb{T}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ is the trajectory of the controlled system. At each time $t \in \mathbb{T}_{\geq 0}$, $\xi(t)$ and $\sigma(t)$ are known by the coder, but only $\sigma(t)$ is known by the decoder (mode-dependent setting). †For each $k \in \mathbb{N}$ and $t \in [kT, (k+1)T) \cap \mathbb{T}$, the coder and the decoder can compute $\chi(kT, (k-1)T, \hat{x}_{k-1}/n_{k-1}, \sigma; \mathsf{SwS}\|\kappa)$, $\mathring{\chi}(kT, (k-1)T, \sigma; \mathsf{SwS}^\circ)$ and $\chi(t, kT, \hat{x}_k/n_k, \sigma; \mathsf{SwS}\|\kappa)$, for instance by integrating auxiliary systems. ‡For each $k \in \mathbb{N}$, the decoder can compute $Q_{\alpha,\bar{A}_k}$ from $\bar{A}_k$, and thus it can compute $\hat{y}_k$ from $e(k)$.

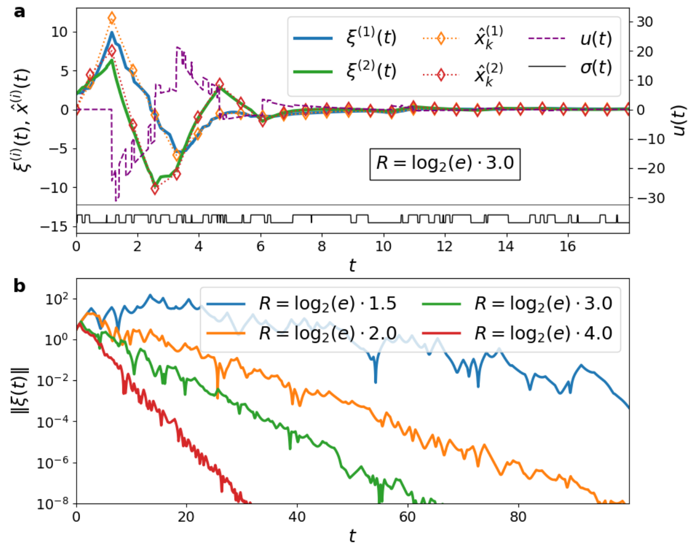


Figure 3.8: **a**: Evolution of ξ and u for a sample execution of the system controlled by the coder–decoder described in Figure 3.7 with data rate $R = 3\log_2(e)$. The black curve below the plot represents the switching signal σ. The orange and red diamonds represent the value of $\hat{x}_k$ at the transmission times $t = kT$, $k \in \mathbb{N}$. **b**: Evolution of the norm of ξ for sample executions of the system controlled by the coder–decoder described in Figure 3.7 with data rates $R \in \log_2(e)\{1.5, 2, 3, 4\}$.

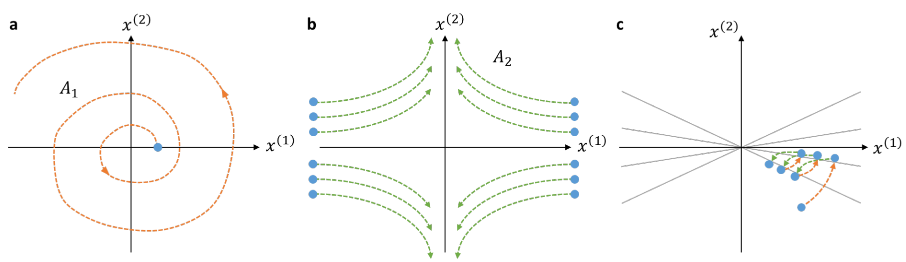


Figure 3.9: Switched linear system of Example 3.6. **a**: Effect of the matrix A_1 ("rotation + divergence"). **b**: Effect of the matrix A_2 ("horizontal convergence + vertical divergence"). **c**: Switching control strategy for the system, with switching signal as control input.

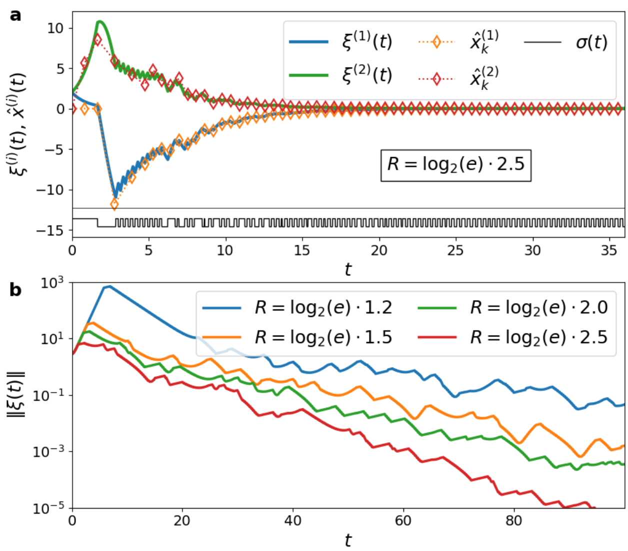


Figure 3.10: **a**: Evolution of ξ and u for a sample execution of the system controlled by the coder–decoder described in Example 3.6 with data rate $R = 2.5 \log_2(e)$. The black curve below the plot represents the switching signal σ. The orange and red diamonds represent the value of $\hat{x}_k$ at the transmission times $t = kT$, $k \in \mathbb{N}$. **b**: Evolution of the norm of ξ for sample executions of the system controlled by the coder–decoder described in Example 3.6 with data rates $R \in \log_2(e)\{1.2, 1.5, 2, 2.5\}$.

Coder

Input: A switched linear system $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$ with dwell time τ_{a}, a static controller $\kappa : \mathbb{R}^n \times \Sigma \to \mathbb{R}^m$ for SwS. A resolution $\alpha > 0$, a period $T \in \mathbb{R}_{>0}$, and a subperiod $p \in \mathbb{N}_{>0}$.

Output: Symbols $e(j)$ to be sent to the decoder at each time $t = jT$, $j \in \mathbb{N}$.

Algorithm:

Let $\hat{x}_0 = 0 \in \mathbb{R}^n$, $r_0 = s_0 = 1$ and $v_0 = 0$

Loop: at time $t = kpT$ **for** $k = 0, 1, \ldots$

- Measure $\xi(kpT)$ and let $y_k = \xi(kpT) - \hat{x}_k$.
- **If** $k > 0$
  - Let $r_k = e^{\nu pT} s_{k-1} + (\bar{a}(v_k) + \bar{b}(v_k)) r_{k-1}$.†
  - Let $s_k = e^{\nu pT} \alpha s_{k-1} + \bar{b}(v_k) r_{k-1}$.†
- Let $\hat{y}_k = Q_\alpha(y_k / s_k)$.
- Encode the pair $(\hat{y}_k, v_k)$ as p symbols $e(kp), e(kp+1), \ldots, e(kp+p-1)$.
- **Loop:** at time $t = (kp+j)T$ **for** $j = 0, 1, \ldots, p-1$
  - Send $e(kp+j)$ and $\sigma((kp+j)T)$ to the decoder.
- Let $\tilde{x}_{k+1} = \chi((k+1)pT, kpT, \hat{x}_k, \hat{\sigma}; \mathsf{SwS}\|\kappa)$.‡
- Let $\tilde{y}_{k+1} = \chi((k+1)pT, kpT, \hat{y}_k, \hat{\sigma}; \mathsf{SwS}^\circ)$.‡
- Let $\hat{x}_{k+1} = \tilde{x}_{k+1} + s_k \tilde{y}_{k+1}$.
- Let $v_{k+1} = N_\sigma^{\mathrm{sw}}(T; kp, (k+1)p)$.†

Decoder

Input: A switched linear system $\mathsf{SwS} \sim (\mathbb{R}^n, \mathbb{R}^m, \{A_i\}_{i\in\Sigma}, \{B_i\}_{i\in\Sigma})$ with dwell time τ_{a}, a static controller $\kappa : \mathbb{R}^n \times \Sigma \to \mathbb{R}^m$ for SwS. A resolution $\alpha > 0$, a period $T \in \mathbb{R}_{>0}$, and a subperiod $p \in \mathbb{N}_{>0}$.

Output: A control input $u(t)$ at each time $t \in \mathbb{R}_{\geq 0}$ that is applied to the system.

Algorithm:

Let $\hat{x}_0 = 0 \in \mathbb{R}^n$, and $r_0 = s_0 = 1$.

Loop: at time $t = kT$ **for** $k = 0, 1, \ldots$

- **Loop:** at time $t = (kp+j)T$ **for** $j = 0, 1, \ldots, p-1$
  - Receive $e(kp+j)$ and $\sigma((kp+j)T)$.
  - For each $t \in [(kp+j)T, (kp+j+1)T)$, let $u(t) = \kappa(\chi(t, kpT, \hat{x}_k, \hat{\sigma}; \mathsf{SwS}\|\kappa), \hat{\sigma}(t))$.‡
- Let $\hat{y}_k$ and v_k be decoded from the previously received symbols.
- **If** $k > 0$
  - Let $r_k = e^{\nu pT} s_{k-1} + (\bar{a}(v_k) + \bar{b}(v_k)) r_{k-1}$.†
  - Let $s_k = e^{\nu pT} \alpha s_{k-1} + \bar{b}(v_k) r_{k-1}$.†
- Let $\tilde{x}_{k+1} = \chi((k+1)pT, kpT, \hat{x}_k, \hat{\sigma}; \mathsf{SwS}\|\kappa)$.‡
- Let $\tilde{y}_{k+1} = \chi((k+1)pT, kpT, \hat{y}_k, \hat{\sigma}; \mathsf{SwS}^\circ)$.‡
- Let $\hat{x}_{k+1} = \tilde{x}_{k+1} + s_k \tilde{y}_{k+1}$.

Figure 3.11: Implementation of the coder–decoder. $(\xi, \sigma) : \mathbb{T}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ is the trajectory of the controlled system. †See (3.13) for the definition of N_σ^{sw}, and (3.14) for the definitions of $\bar{a}$ and $\bar{b}$. ‡$\hat{\sigma}$ is the sample-and-hold version of σ with period T, built from the observations of σ by the coder at times $t = jT$, $j \in \mathbb{N}$ (see paragraph "Coder and decoder implementations").

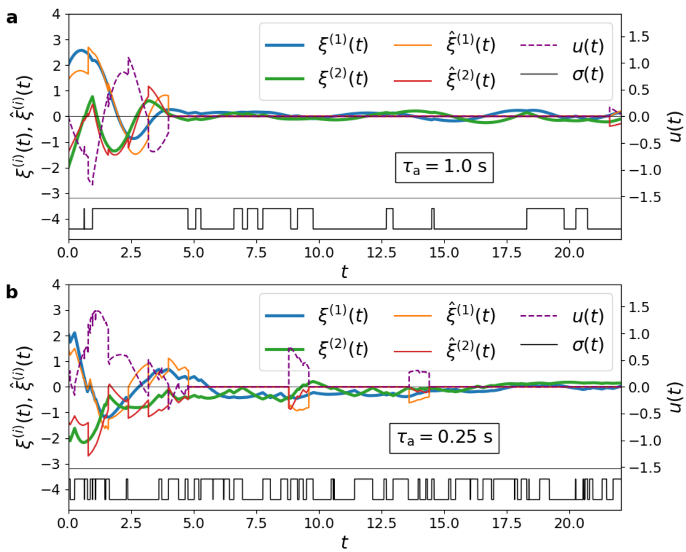


Figure 3.12: Evolution of ξ and u for a sample execution of the system controlled by the coder–decoder described in Figure 3.11, for different values of the average dwell time. The black curve below the plot represents the switching signal σ. The orange and red curves represent the evolution of $\hat{\xi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$, defined by $\hat{\xi}(t) = \chi(t, kT, \hat{x}_k, \hat{\sigma}; \mathsf{SwS} \| \kappa)$ for each $t \in [kpT, (k+1)pT)$, $k \in \mathbb{N}$.

Conclusions

The goal of systems and control theory is to study natural phenomenons (e.g., biological processes), technological devices (e.g., robots) or combinations of both (e.g., pacemakers), and to design strategies to control them so that they behave in some specified way. For that, we rely on mathematical models describing the evolution of these phenomenons/devices (called **systems**) and describing how they react to external inputs. The challenge with modern systems is that these systems are becoming immensely complex and have many non-standard characteristics. We think for instance to cyber-physical systems which involve the interaction of discrete and continuous dynamics (called **hybrid behavior**) and often include spatially distributed components that communicate through a shared communication network (called **networked systems**). These characteristics preclude the use of classical control techniques and thus call for the development of new mathematical and algorithmic tools to address these fundamental challenges of modern control systems and to provide workable solutions for emerging concrete control problems.

In this thesis, we studied these two fundamental and challenging aspects of modern control systems (hybrid behavior and networked systems) from the perspective of **switched systems** (a paradigmatic class of hybrid systems) and **quantized control** (control with quantization errors and limited information flow). For that, we focused on the property that the dynamics of theses systems can often be divided into several distinct components, which grow at different speeds (sometimes referred to as *slow and fast modes separation*).

First, we studied the property of separation of the dynamics, from the theoretical and algorithmic points of view, for switched linear systems and for nonlinear smooth dynamical systems; see Chapter 2. Therefore, we leveraged several classical tools from systems and control theory, like the concepts of hyperbolicity and positivity, and extended them for the study of switched linear systems. In particular, the concept of **hyperbolicity** was very useful as it accounts for the property that the linearized dynamics of a dynamical system can be decomposed into a "dominant" component that grows exponentially fast and a "dominated" component that converges exponentially to zero. We extended this property to switched linear systems, and we provided an algorithmic framework for the verification of this property, drawing on advanced tools from applied mathematics, like conic optimization and automata theory. We also considered a more general notion of separation of the dynamics in which the dominated component grows exponentially slower than the dominant dynamics, but does not necessarily converge to zero. Finally, by combining the results for the analysis of this property for switched linear systems with tools from symbolic control (allowing to abstract a nonlinear dynamical system as a collection of "local" linear systems), we provided an algorithmic framework for the study of the property of separation of the linearized dynamics for smooth dynamical systems. We also described several applications of this property, encompassing the convergence of the trajectories of switched linear systems to a low-dimensional time-varying attractor, the study of the robustness of attractors of nonlinear dynamical systems, and the computation of the topological entropy of switched linear systems and nonlinear smooth dynamical systems.

Then, we leveraged the property of separation of the dynamics into components with different rates of growth for the quantized control of switched linear systems; see Chapter 3. Indeed, dynamics with different growth rates generally requires different levels of quantization and different information rates to be described accurately. We used this observation to obtain sharp bounds on the minimal data rate necessary to control or estimate the state of a switched linear system under several communication structures, and for the implementation of associated optimal quantizing–controlling strategies. More precisely,

- In Section 3.2, we studied the question of **the minimal data rate for state estimation of linear time-varying systems**. We established the equivalence between this quantity and the topological entropy of the system, thereby extending the classical "data rate theorems" of time-invariant systems to this class of time-varying systems. We also improved some existing results regarding the computation of the topological entropy of linear time-varying systems, using the theory of p-dominance developed in the first part of the thesis.

- In Section 3.3, we introduced the concept of **worst-case topological entropy of switched linear systems**. We showed that this concept was key for the theoretical analysis and practical computation of quantized control strategies when the mode of the system is known in real time by the coder–decoder (**mode-dependent setting**). Therefore, we leveraged several tools from applied mathematics, namely the concepts of exterior algebras and Joint Spectral Radius, and tools developed in the first part of the thesis, to study the separation of the growth rate of the different components of the dynamics.

- In Section 3.4, we provided theoretical and practical insights on what can and cannot be achieved with respect to the quantized control of switched linear systems when the mode is not known in real time by the decoder (**mode-oblivious setting**). We showed that, in this setting, switched systems are in general impossible to *observe* with a finite data rate and we demonstrated the importance of **switching signal's trackability** for the stabilizability (i.e., *control*) of these systems under data-rate constraints.

Directions for future research

The above suggests several interesting directions for future research.

Firstly, it would be worth studying generalizations of the property of separation of the dynamics for other classes of hybrid systems, like switched linear systems with state-dependent switching and nonlinear hybrid systems. It would also be interesting to investigate further applications of the property of hyperbolicity in modern control problems, for instance, for the study of the dimension of attractors of hybrid systems or to perform dimensionality reduction in algorithmic control problems (such as abstractions computation, safety analysis, and controller synthesis), as we did for the computation of the worst-case topological entropy of switched linear systems.

Secondly, the result on the equivalence of topological entropy and the minimal data rate for state estimation of LTV systems gives rise to several interesting open questions. For instance, it would be worth investigating whether we can generalize this result to nonlinear time-varying systems, and if yes, under which assumptions. Also, we saw that the resulting coder–decoder requires in general an unbounded memory. It would be interesting to inquire variants of the notion of topological entropy accounting for the minimal data rate for state estimation of LTV systems with a coder–decoder with finite memory.

Thirdly, a straightforward declination of the mode-dependent setting for the quantized control of switched linear systems is the quantized control of these systems when the mode of the system can be controlled by the coder–decoder. We already provided a proof-of-concept that the notion of worst-case topological entropy can be used for this kind of problems (Example 3.6 in Subsection 3.3.4), but it would be interesting to investigate how our results could be particularized for this specific application; for instance, whether we can improve the bounds on the minimal data rate for stabilization when the decoder can choose the mode of the system (e.g., we could use the selection of the mode to reduce the necessary data rate), and whether we can derive a computable closed-form expression for the optimal data rate.

Finally, the results on the quantized stabilizability of switched linear systems in the mode-oblivious setting call for several further research directions. Indeed, for the moment, there is still a gap between the sufficient conditions and the necessary conditions for these systems to be stabilizable with a finite data rate. More precisely, we provided a sufficient condition on the switching signal (namely, that it has a nonzero average dwell time) to ensure stabilizability of the system with a finite data rate, and we showed with examples that this condition cannot be removed in general (that is, without adding other assumptions). It would be interesting, from the theoretical and practical points of view, to derive sufficient and necessary conditions for the stabilizability of these systems under data-rate constraints, in the mode-oblivious setting. Furthermore, this setting also allows to address the quantized control of other classes of hybrid systems, like switched linear systems with state-dependent switching and hybrid linear systems. It would nevertheless be worth investigating whether the coder–decoder and its data rate could be improved for these other classes of systems, possibly considering other assumptions on the system (for instance, in the case of state-dependent switching, an abrupt change in the state can be an evidence that a switching has occured and this information could be used to improve the data rate of the coder–decoder).

The questions of switching and control under communication constraints are already fundamental for a wide range of modern control problems, and they are likely to become even more important in the coming years due the phenomenal outbreak of cyber-physical systems and communication technologies of all kinds. This will require, among others, formal verification tools where these two aspects of modern control systems are integrated. In this thesis, we aimed to lay the basis for the development of such tools, by developing theoretical and computational frameworks for the analysis of switched linear systems and for their control under communication constraints. In particular, we showed that the analysis and control of these systems can benefit from recent advanced technologies developed within the fields of mathematics, computer science and optimization. This provides proof-of-concepts that a multidisciplinary approach is essential for the development of such formal verification tools for modern control problems, and we hope that the present work will stimulate the research in that direction for the coming years.

Appendix

A.1 Proofs of Chapter 1

A.1.1 Proof of Proposition 1.16

Let α_1 and α_2 be class-$\mathcal{K}$ functions as in Definition 1.14 for the equivalence of $\mathfrak{C}_1$ and $\mathfrak{C}_2$. Assume that HySys is stable with respect to $\mathfrak{C}_1$ and let $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a class-$\mathcal{KL}$ function as in Definition 1.15 for the stability of HySys with respect to $\mathfrak{C}_1$. Let ϕ be a trajectory of HySys and fix $t \in \mathbb{R}_{\geq 0}$. It holds that $\mathfrak{C}_2(\phi(t)) \leq \alpha_2(\mathfrak{C}_1(\phi(t))) \leq \alpha_2(\beta(\mathfrak{C}_1(\phi(t)), t)) \leq \alpha_2(\beta(\alpha_1(\mathfrak{C}_2(\phi(t))), t))$. It is readily checked that the function $(r, t) \mapsto \alpha_2(\beta(\alpha_1(r), t))$ is of class $\mathcal{KL}$, showing that HySys is stable with respect to $\mathfrak{C}_2$. The proof that HySys is stable with respect to $\mathfrak{C}_1$ if it is stable with respect to $\mathfrak{C}_2$ is identical. This concludes the proof of the proposition. □

A.1.2 Proof of Proposition 1.28

We show that Sys is stable with respect to V. By Proposition 1.16, this will imply that Sys is stable with respect to $\mathfrak{C}$. Let $T \in \mathbb{T}_{>0}$ and $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a class-$\mathcal{K}$ function as in Definition 1.26 for the Lyapunov function V. Let ξ be a trajectory of Sys and fix $t \in \mathbb{T}_{\geq 0}$. By Proposition 1.27, it holds that $0 \leq V(\xi(t)) \leq V(\xi(t')) \leq V(\xi(0))$ for all $t' \in [0, t] \cap \mathbb{T}$. Hence, $\alpha(V(\xi(kT))) \geq \alpha(V(\xi(t)))$ for all $k \in \{0, \ldots, \lfloor t/T \rfloor\}$, where $\lfloor \cdot \rfloor$ is the *floor* function. From the definition of T and α, it follows that $\alpha(V(\xi(t)))\lfloor t/T \rfloor \leq V(\xi(0))$, so that $V(\xi(t)) \leq \alpha^{-1}(\min(\alpha(V(\xi(0))), V(\xi(0))/\lfloor t/T \rfloor))$ where α^{-1} is the inverse of α. It is readily checked that the function $(r, t) \to \alpha^{-1}(\min(\alpha(r), r/\lfloor t/T \rfloor))$ is of class-$\mathcal{KL}$, concluding the proof of the proposition. □

A.1.3 Proof of Proposition 1.51

We provide a proof only for the case of continuous-time systems since the case of discrete-time systems is similar and simpler. We will show that $1 \Rightarrow 4$ and

$3 \Rightarrow 1$. The implications $1 \Rightarrow 4 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$ and $1 \Rightarrow 4 \Rightarrow 5 \Rightarrow 3 \Rightarrow 1$ will then follow trivially and this will conclude the proof for the continuous-time case.

In order to prove the implication $1 \Rightarrow 4$, assume that $\hat{\varrho}(\mathsf{SwS}) < 0$. Then, there is $T \in \mathbb{T}_{>0}$ such that $\frac{1}{T}\log(\sup\{\|\mathring{\chi}(T,0,\sigma)\| : \sigma \in \mathcal{S}\}) < 0$. Hence, it holds that $\sup\{\|\mathring{\chi}(T,0,\sigma)\| : \sigma \in \mathcal{S}\} \leq \rho < 1$. By Proposition 1.39, it follows that any trajectory $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of HySys satisfies $\|\phi(jT+T)\| \leq \rho\|\phi(jT)\|$ for all $j \in \mathbb{N}$. This implies for all $j \in \mathbb{N}$, $\|\phi(jT)\| \leq \rho^j\|\phi(0)\|$, showing the HySys is GES.

In order to prove the implication $3 \Rightarrow 1$, we proceed by contradiction. Therefore, assume that $\hat{\varrho}(\mathsf{SwS}) \geq 0$ and let $\mathcal{M} = \{\mathring{\chi}(1,0,\sigma) \in \mathbb{R}^{n\times n} : \sigma \in \mathcal{S}\}$. It holds that $\lim_{k\to\infty}\frac{1}{k}\log(\sup\{\|M_1 \dots M_k\| : M_1,\dots,M_k \in \mathcal{M}\}) = \hat{\varrho}(\mathsf{SwS}) \geq 0$. The first quantity in the previous equation is the joint spectral radius of a "discrete-time switched system" under arbitrary switching with set of matrices $\mathcal{M}$ (the set $\mathcal{M}$ is infinite but the principle is the same). Since $\mathcal{M}$ is bounded, it holds by classical results on the joint spectral radius of discrete-time switched systems (see, e.g., Jungers, 2009, Theorem 1.2) that there is an infinite sequence of matrices $(M_j)_{j=1}^J \subseteq \mathcal{M}$ such that $\limsup_{j\to\infty}\|M_jM_{j-1}\cdots M_1\| > 0$. The sequence $(M_j)_{j=1}^J$ defines a switching signal $\sigma \in \mathcal{S}$ satisfying that $M_jM_{j-1}\cdots M_1 = \mathring{\chi}(j,0,\sigma)$ for all $j \in \mathbb{N}$. It follows that $\limsup_{t\to\infty}\mathring{\chi}(t,0,\sigma) > 0$, so that there is $x \in \mathbb{R}^n$ such that $\chi(t,x,\sigma) \not\to 0$ as $t \to \infty$. This is a contradiction with Item 3, concluding the proof that $3 \Rightarrow 1$. □

A.1.4 Proof of Proposition 1.64

Let $x_0 \in \mathbb{R}^n$, $r_0 \geq 0$ and $v \in [-r_0,r_0]^n$. From the assumption on χ, it holds that $\|\chi(T,x_0+v) - \chi(T,x_0) - \frac{\partial\chi}{\partial x}(T,x_0)v\|_\infty \leq \frac{L}{2}\|v\|_\infty^2 \leq \frac{L}{2}r_0^2$ (see, e.g., Berger et al., 2020, Theorem 4.1 with $\nu = 1$). Hence, $\chi(T,x_0+v) - \chi(T,x_0) - \frac{\partial\chi}{\partial x}(T,x_0)v \in \frac{L}{2}[-r_0^2,r_0^2]^n$, and thus, since $v \in [-r_0,r_0]^n$, it follows that $\chi(T,x_0+v)-\chi(T,x_0) \in \frac{\partial\chi}{\partial x}(T,x_0)[-r_0,r_0]^n+\frac{L}{2}[-r_0^2,r_0^2]^n$, which concludes the proof. □

A.1.5 Proof of Corollary 1.65

Let $x_0,x_1 \in \mathbb{R}^n$ and $r_0,r_1 \geq 0$. By Proposition 1.64, it holds that if $\{\chi(T,x) : x \in x_0+[-r_0,r_0]^n\}\cap(x_1+[-r_1,r_1]^n) \neq \emptyset$, then $\chi(T,x_0)+\frac{\partial\chi}{\partial x}(T,x_0)[-r_0,r_0]^n + \frac{L}{2}[-r_0^2,r_0^2]^n \cap (x_1+[-r_1,r_1]^n) \neq \emptyset$. The latter is equivalent to saying that $x_1 - \chi(T,x_0) \in \frac{\partial\chi}{\partial x}(T,x_0)[-r_0,r_0]^n + \frac{L}{2}[-r_0^2,r_0^2]^n - [-r_1,r_1]^n$. The conclusion then follows from $\frac{L}{2}[-r_0^2,r_0^2]^n - [-r_1,r_1]^n = [-r_1-\frac{L}{2}r_0^2, r_1+\frac{L}{2}r_0^2]^n$. □

A.2 Proofs of Chapter 2

A.2.1 Results from linear algebra

Notation. For $P \in \mathbb{S}^{n\times n}$, we let $\nu(P)$ be the number of negative eigenvalues of P, and $\nu_0(P)$ the number of nonpositive eigenvalues of P.

Theorem A.40 (Sylvester inertia theorem, Horn and Johnson, 1985, Section 4.5)**.** *Let $Q = A^\top PA$ where $P \in \mathbb{S}^{n\times n}$ and $A \in \mathbb{R}^{n\times n}$. Then, $\nu(Q) \leq \nu(P)$ and $\nu_0(Q) \geq \nu_0(P)$.*[17]

Theorem A.41 (Min-max principle, Horn and Johnson, 1985, Section 4.2)**.** *Let $P \in \mathbb{S}^{n\times n}$ and $k \in \{0,\ldots,n\}$. Then, $\nu(P) \geq k$ (resp. $\nu_0(P) \geq k$) if and only if there is a subspace $H \subseteq \mathbb{R}^n$ with dimension k such that $x^\top Px < 0$ (resp. ≤ 0) for all $x \in H \setminus \{0\}$.*

Theorem A.42 (Main inertia theorem, Lancaster and Tismenetsky, 1985, Section 13.2)**.** *For any matrix $A \in \mathbb{R}^{n\times n}$, there is a matrix $P \in \mathbb{S}^{n\times n}$ satisfying $A^\top PA - P \prec 0$ if and only if A has no eigenvalue with modulus $|\lambda_i| = 1$. Moreover, in this case, $P \in \mathbb{S}_p^{n\times n}$ where p is the number of eigenvalues of A with modulus $|\lambda_i| > 1$.*

A.2.2 Proof of Proposition 2.3

Item 1 follows from Theorem A.42. The equivalence of Items 1 and 2 follows directly from the eigenvalue decomposition of A. □

A.2.3 Proof of Proposition 2.5

The "only if" direction is straightforward from the definition of $\mathcal{K}(P)$ and the observation that $\operatorname{int}\mathcal{K}(P) = \{x \in \mathbb{R}^n : x^\top Px < 0\}$. For the "if" direction, observe that (2.3) implies that for every $x \in \mathbb{R}^n \setminus \{0\}$ such that $x^\top Px \leq 0$, it holds that $x^\top A^\top P\,Ax < 0$. Therefrom, we deduce the dissipation inequality (2.1) by applying the $\mathcal{S}$-Lemma (see, e.g., Ben-Tal and Nemirovski, 2001, Theorem 4.3.3, or Boyd and Vandenberghe, 2004, Section B.2). □

[17] The proof in Horn and Johnson (1985, Section 4.5) is presented for A invertible with the conclusion that $\nu(Q) = \nu(P)$ and $\nu_0(Q) = \nu_0(P)$. The case of A singular follows by applying a small perturbation on A and using the continuous dependence of the eigenvalues of symmetric matrices.

A.2.4 Proof of Theorem 2.11

Part 1: $1 \Rightarrow 2$

Assume that SwS is p-dominant with $\mathsf{Aut} = (Q, \Sigma, \Theta)$, $\{\gamma_\theta\}_{\theta\in\Theta}$ and $\{P_q\}_{q\in Q}$ and let $\epsilon > 0$ be such that the right-hand side of the dissipation inequalities (2.4) can be replaced by $-\epsilon I$. Fix $\sigma \in \mathcal{S}$. We will build a p-splitting $(E^{\mathrm{s}}_\sigma, E^{\mathrm{u}}_\sigma)$ which satisfies (2.6) for some $C \geq 1$ and $\mu \in (0, 1)$ independent of σ.

Therefore, let $(\theta_t)_{t=0}^{\infty}$ be a path in Aut such that $\sigma(t) = \mathsf{i}(\theta_t)$ for all $t \in \mathbb{N}$. For each $q \in Q$, let $V_q(x) = x^\top P_q x$. Remember that (2.4) implies that for all $t \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$V_{\mathsf{s}(\theta_{t+1})}(A_{\sigma(t)}x) \leq \gamma^2_{\theta_t} V_{\mathsf{s}(\theta_t)}(x) - \epsilon\|x\|^2. \tag{A.15}$$

The component $E^{\mathrm{u}}_\sigma : \mathbb{N} \rightrightarrows \mathbb{R}^n$ is defined as follows. Let $E^{\mathrm{u}}_\sigma(0)$ be any p-dimensional subspace of $\mathbb{R}^n$ satisfying $x \in E^{\mathrm{u}}_\sigma(0) \Rightarrow V_{\mathsf{s}(\theta_0)}(x) \leq 0$ (see Theorem A.41). Then, define E^{u}_σ on $\mathbb{N}_{>0}$ as follows: $E^{\mathrm{u}}_\sigma(t) = \mathring{\chi}(t, 0, \sigma)E^{\mathrm{u}}_\sigma(0)$ for all $t \in \mathbb{N}$ where $\mathring{\chi}$ is the fundamental matrix solution[18] of SwS. By (A.15), it holds that for every $t \in \mathbb{N}_{>0}$ and $x \in E^{\mathrm{u}}_\sigma(0) \setminus \{0\}$, $V_{\mathsf{s}(\theta_t)}(\mathring{\chi}(t, 0, \sigma)x) < 0$. This implies that for all $t \in \mathbb{N}$, $\operatorname{Ker} \mathring{\chi}(t, 0, \sigma) \cap E^{\mathrm{u}}_\sigma(0) = \{0\}$, so that $\dim E^{\mathrm{u}}_\sigma(t) = \dim E^{\mathrm{u}}_\sigma(0) = p$.

The dominated component $E^{\mathrm{s}}_\sigma : \mathbb{N} \rightrightarrows \mathbb{R}^n$ is defined as follows. For each $t_0, t_1 \in \mathbb{N}$, $t_1 > t_0$, let $E'_{t_1,t_0} = \{x \in \mathbb{R}^n : V_{\mathsf{s}(\theta_{t_1})}(\mathring{\chi}(t_1, t_0, \sigma)x) \geq 0\}$ and define $E^{\mathrm{s}}_\sigma(t_0) = \bigcap_{t_1>t_0} E'_{t_1,t_0}$. We will show that for each $t_0 \in \mathbb{N}$, $E^{\mathrm{s}}_\sigma(t_0)$ contains at least one linear subspace with dimension $n-p$; the fact that $E^{\mathrm{s}}_\sigma(t_0)$ is actually a linear subspace with dimension $n-p$ will be obtained at the end of this proof. By Theorem A.40, it holds that $\mathring{\chi}(t_1, t_0, \sigma)^\top P_{\mathsf{s}(\theta_{t_1})}\mathring{\chi}(t_1, t_0, \sigma)$ has at least $n-p$ nonnegative eigenvalues; thus by Theorem A.41, E'_{t_1,t_0} contains at least one linear subspace with dimension $n-p$. Moreover, (A.15) implies that E'_{t_1,t_0} is decreasing with respect to t_1: for all $t_0, t_1, t_2 \in \mathbb{N}$, $t_2 > t_1 > t_0$, $E'_{t_2,t_0} \subseteq E'_{t_1,t_0}$. Hence, with a standard compactness argument (see, e.g., Berger et al., 2018, Lemma 7), it follows that for each $t_0 \in \mathbb{N}$, the intersection $\bigcap_{t_1>t_0} E'_{t_1,t_0}$ also contains a subspace of dimension $n-p$.

Now, we show that the pair $(E^{\mathrm{s}}_\sigma, E^{\mathrm{u}}_\sigma)$ defined above satisfies the relation (2.6) for some $C \geq 1$ and $\mu \in (0, 1)$. We will need the following lemma (the proof is presented at the end of this subsection).

Lemma A.43. *Let* Aut*,* $\{\gamma_\theta\}_{\theta\in\Theta}$ *and* $\{P_q\}_{q\in Q}$ *be as above. There is* $\mu \in (0, 1)$ *such that for every* $\theta \in \Theta$*,* $V_{\mathsf{t}(\theta)}(A_{\mathsf{i}(\theta)}x) \leq \gamma^2_\theta \cdot \min\left\{\mu V_{\mathsf{s}(\theta)}(x), \frac{1}{\mu}V_{\mathsf{s}(\theta)}(x)\right\}$.

Let μ be as in Lemma A.43, and let $K = \max_{q\in Q} \|P_q\|$. Let $t_0 \in \mathbb{N}$ and $x_1 \in E^{\mathrm{s}}_\sigma(t_0) \setminus \{0\}$. Then, by definition of $E^{\mathrm{s}}_\sigma(t_0)$, Lemma A.43 and (A.15), it

[18] For a reminded, see Definition 1.47 in Subsection 1.3.2.

holds that for every $t_1 \in \mathbb{N}$, $t_1 > t_0$,

$$\begin{aligned}\epsilon\|\chi(t_1,t_0,x_1,\sigma)\|^2 \le \gamma_{\theta_{t_1}}^2 V_{\mathsf{s}(\theta_{t_1})}(\chi(t_1,t_0,x_1,\sigma)) &\le \gamma_{\theta_{t_1}}^2\gamma_{\theta_{t_1-1}}^2 \mu V_{\mathsf{s}(\theta_{t_1-1})}(\chi(t_1-1,t_0,x_1,\sigma))\\ &\le (\gamma_{\theta_{t_0}}\cdots\gamma_{\theta_{t_1}})^2\mu^{t_1-t_0}V_{\mathsf{s}(\theta_{t_0})}(x_1) \le (\gamma_{\theta_{t_0}}\cdots\gamma_{\theta_{t_1}})^2\mu^{t_1-t_0}K\|x_1\|^2.\end{aligned} \tag{A.16}$$

Similarly, if $x_2 \in E_\sigma^{\mathrm{u}}(t_0)\setminus\{0\}$, then for all $t_1 \in \mathbb{N}$, $t_1 > t_0$,

$$\begin{aligned}-K\|\chi(t_1,t_0,x_2,\sigma)\|^2 \le V_{\mathsf{s}(\theta_{t_1})}(\chi(t_1,t_0,x_2,\sigma)) &\le \gamma_{\theta_{t_1-1}}^2 \tfrac{1}{\mu} V_{\mathsf{s}(\theta_{t_1-1})}(\chi(t_1-1,t_0,x_2,\sigma))\\ &\le (\gamma_{\theta_{t_0+1}}\cdots\gamma_{\theta_{t_1-1}})^2\mu^{t_0-t_1+1}V_{\mathsf{s}(\theta_{t_0+1})}(\chi(t_0+1,t_0,x_2,\sigma))\\ &\le -(\gamma_{\theta_{t_0+1}}\cdots\gamma_{\theta_{t_1-1}})^2\mu^{t_0-t_1+1}\epsilon\|x_2\|^2.\end{aligned}$$

Taking the quotient of $\|\chi(t_1,t_0,x_1,\sigma)\|$ and $\|\chi(t_1,t_0,x_2,\sigma)\|$, it follows that (2.6) holds with μ as above and with $C = \epsilon^{-1}K\frac{1}{\sqrt{\mu}}\max_{\theta\in\Theta}\gamma_\theta^2$. In particular, μ and C are independent of σ and t_0. Since t_0 is arbitrary, this holds true for every $t_0 \in \mathbb{N}$.

Finally, we use (2.6) to show that for all $t \in \mathbb{N}$, $E_\sigma^{\mathrm{s}}(t)$ is a linear subspace with dimension $n-p$. Therefore, fix $t_0 \in \mathbb{N}$ and assume that $\dim(\operatorname{span} E_\sigma^{\mathrm{s}}(t_0)) > n-p$. Then, $\operatorname{span} E_\sigma^{\mathrm{s}}(t_0)\cap E_\sigma^{\mathrm{u}}(t_0) \neq \{0\}$, so there is $x \in E_\sigma^{\mathrm{u}}(t_0)\setminus\{0\}$ and $x_1, x_2 \in E_\sigma^{\mathrm{s}}(t_0)$ such that $x = x_1 + x_2$. It follows that for every $t_1 \in \mathbb{N}$, $t_1 \ge t_0$, $\|\chi(t_1,t_0,x,\sigma)\| \le 2\max\{\|\chi(t_1,t_0,x_1,\sigma)\|, \|\chi(t_1,t_0,x_2,\sigma)\|\}$; a contradiction with (2.6). Hence, $E_\sigma^{\mathrm{s}}(t_0)$ is a linear subspace with dimension $n-p$, concluding the proof that $1 \Rightarrow 2$. □

Proof of Lemma A.43. Because $\{P_q\}_{q\in Q}$ is finite there is $\alpha > 0$ such that, for every $q \in Q$ and every $x \in \mathbb{R}^n$, $-\epsilon I \preceq \alpha P_q \preceq \epsilon I$. Hence, the right-hand side of (2.4) can be replaced by αP_{q_1} or $-\alpha P_{q_1}$. This concludes the proof, since by the finiteness of Θ, there is $\mu \in (0,1)$ such that for every $\theta \in \Theta$, $\mu\gamma_\theta^2 \le \gamma_\theta^2 - \alpha < \gamma_\theta^2 + \alpha \le \mu^{-1}\gamma_\theta^2$. □

Part 2: $2 \Rightarrow 1$

Assume that SwS satisfies Item 2 with $C \ge 1$, $\mu \in (0,1)$ and dominated p-splitting $(E_\sigma^{\mathrm{s}}, E_\sigma^{\mathrm{u}})$ for each $\sigma \in \mathcal{S}$. The proof that $2 \Rightarrow 1$ relies on the following technical lemma (see Berger and Jungers, 2019, Lemma 6, for a proof).

Lemma A.44. *Let* SwS *and* $(E_\sigma^{\mathrm{s}}, E_\sigma^{\mathrm{u}})$*, for each* $\sigma \in \mathcal{S}$*, be as above. There is* $c > 0$ *such that for every* $\sigma \in \mathcal{S}$*,* $t \in \mathbb{N}$ *and* $x \in E_\sigma^{\mathrm{u}}(t)$*, it holds that* $\|A_{\sigma(t)}x\| \ge c\|x\|$.

In the following, it will be convenient to describe the decompositions of $\mathbb{R}^n$ induced by the dominated p-splittings with *projection matrices*. More precisely, for each $\sigma \in \mathcal{S}$, we let $R_\sigma : \mathbb{N} \to \mathbb{R}^{n\times n}$ be defined by $R_\sigma(t)$ is the projection on $E^{\mathrm{u}}_\sigma(t)$ parallel to $E^{\mathrm{s}}_\sigma(t)$. Note that for each $\sigma \in \mathcal{S}$ and $t \in \mathbb{N}$, $R_\sigma(t)$ determines $E^{\mathrm{s}}_\sigma(t)$ and $E^{\mathrm{u}}_\sigma(t)$ completely since $\operatorname{Im} R_\sigma(t) = E^{\mathrm{u}}_\sigma(t)$ and $\operatorname{Ker} R_\sigma(t) = E^{\mathrm{s}}_\sigma(t)$ (in particular, it follows that $\operatorname{rank} R_\sigma(t) = p$). The following proposition, which is a straightforward consequence of Lemma A.44 (see Berger and Jungers, 2019, Proposition 7, for a proof), states that the matrices $R_\sigma(t)$ are uniformly bounded for all $\sigma \in \mathcal{S}$ and all $t \in \mathbb{N}$.

Proposition A.45. *Let* SwS *and* R_σ*, for each* $\sigma \in \mathcal{S}$*, be as above. There is* $M \geq 0$ *such that for every* $\sigma \in \mathcal{S}$ *and* $t \in \mathbb{N}$*,* $\|R_\sigma(t)\| \leq M$*.*

Using the above definitions and results, we will build an automaton, a set of rates and a set of symmetric matrices for which SwS is p-dominant. Therefore, fix $T \in \mathbb{N}_{>0}$ such that $C\mu^T \leq \frac{1}{4}$ and fix $r \in (0, \frac{3}{10})$. Let $\mathcal{R}_M$ be the set of all projection matrices $R \in \mathbb{R}^{n\times n}$ of rank p and with $\|R\| \leq M$, where M is as in Proposition A.45: i.e., $\mathcal{R}_M = \{R \in \mathbb{R}^{n\times n} : R^2 = R, \operatorname{rank} R = p, \|R\| \leq M\}$. Since $\mathcal{R}_M$ is bounded, there is a *finite* set $\{S_1, \ldots, S_m\} \subseteq \mathcal{R}_M$ that is an "r-cover" of $\mathcal{R}_M$, meaning that for any $R \in \mathcal{R}_M$, there is $q \in \{1, \ldots, m\}$ such that $\|R - S_q\| \leq r$.

Now, using this set $\{S_1, \ldots, S_m\}$, we build an automaton $\mathsf{Aut}^* = (Q^*, \Sigma^T, \Theta^*)$ and a set of matrices $\{P^*_q\}_{q\in Q^*} \subseteq \mathbb{S}^{n\times n}_p$ as follows. The alphabet of Aut^* is $\Sigma^T = \{(i_1, \ldots, i_T) : i_k \in \Sigma\}$ (the set of words of length T over Σ). The set of states of Aut is defined by $Q^* = \{1, \ldots, m\}$. Then, for each $q \in Q^*$, we let $P^*_q = -S_q^\top S_q + (I - S_q)^\top (I - S_q) = I - S_q - S_q^\top$. By construction, for each $q \in Q^*$, P^*_q is symmetric, and moreover, P^*_q is negative definite on $\operatorname{Im} S_q$ and positive definite on $\operatorname{Ker} S_q$. Hence, by Theorem A.41, $P^*_q \in \mathbb{S}^{n\times n}_p$ for all $q \in Q^*$. Finally, we define the set $\Theta^* \subseteq Q^* \times \Sigma^T \times Q^*$ of admissible transitions in Aut^* as follows: for every $w = (i_1, \ldots, i_T) \in \Sigma^T$ and $q_1, q_2 \in Q^*$, we let $\theta \doteq (q_1, w, q_2) \in \Theta^*$ if and only if there is $\gamma^*_\theta > 0$ such that $\bar{A}_w^\top P^*_{q_2} \bar{A}_w - (\gamma^*_\theta)^2 P^*_{q_1} \prec 0$, where $\bar{A}_w = A_{i_T} \cdots A_{i_1}$.

We show that every $\sigma \in \mathcal{S}$ can be read as the concatenation of words obtained from a path in Aut^*. Therefore, fix $\sigma \in \mathcal{S}$ and decompose σ into words of length T: that is, for every $t \in \mathbb{N}$, let $w_t = (\sigma(tT), \ldots, \sigma(tT + T - 1)) \in \Sigma^T$. Then, for each $t \in \mathbb{N}$, let $q_t \in Q^*$ be such that $\|R_\sigma(tT) - S_{q_t}\| \leq r$, which always exists since $\|R_\sigma(tT)\| \leq M$ (Proposition A.45). We claim that for all $t \in \mathbb{N}$, $(q_t, w_t, q_{t+1}) \in \Theta^*$, which would prove the assertion at the beginning of the paragraph. To prove this claim, we fix $t \in \mathbb{N}$, and we will show that there is $\gamma > 0$ such that $\bar{A}_{w_t}^\top P^*_{q_{t+1}} \bar{A}_{w_t} - \gamma^2 P^*_{q_t} \prec 0$, where $\bar{A}_{w_t} = A_{\sigma(tT+T-1)} \cdots A_{\sigma(tT)}$.

Indeed, let γ be any positive real satisfying

$$2\max\{\,\|\bar{A}_{w_t}x\| : x\in E^{\mathrm{s}}_\sigma(tT),\ \|x\|=1\,\}\le\gamma\le\tfrac{1}{2}\min\{\,\|\bar{A}_{w_t}x\| : x\in E^{\mathrm{u}}_\sigma(tT),\ \|x\|=1\,\}. \tag{A.17}$$

The existence of γ is ensured by (2.6) and $C\mu^T\le\frac{1}{4}$. Also, Lemma A.44 ensures that the right-hand side of (A.17) is positive, so that we can always choose $\gamma>0$. To show that $\bar{A}_{w_t}^\top P^*_{q_{t+1}}\bar{A}_{w_t}-\gamma^2P^*_{q_t}\prec 0$, we let $x\in\mathbb{R}^n\setminus\{0\}$ and $y=\bar{A}_{w_t}x$, and we will show that $y^\top P^*_{q_{t+1}}y<\gamma^2x^\top P^*_{q_t}x$. Therefore, let $x_1\in E^{\mathrm{u}}_\sigma(tT)$ and $x_2\in E^{\mathrm{s}}_\sigma(tT)$ such that $x=x_1+x_2$, and let $y_1=\bar{A}_{w_t}x_1$ and $y_2=\bar{A}_{w_t}x_2$. Then, since $\|R_\sigma(tT)-S_{q_t}\|\le r$ and $\|R_\sigma(tT+T)-S_{q_{t+1}}\|\le r$, we get from the definition of $\{P_q\}_{q\in Q^*}$, the following relations (we use capital letters, X_1,X_2,Y_1,Y_2, to denote the norm of the related vectors; e.g., $X_1=\|x_1\|$):

$$x^\top P^*_{q_t}x\ge -X_1^2+X_2^2-2r(X_1^2+X_2^2),\quad\text{and}\quad y^\top P^*_{q_{t+1}}y\le -Y_1^2+Y_2^2+2r(Y_1^2+Y_2^2).$$

We also have the relations $Y_1\ge 2\gamma X_1$ and $Y_2\le\frac{1}{2}\gamma X_2$ from (A.17). Hence,

$$\begin{aligned}\gamma^{-2}y^\top P^*_{q_{t+1}}y-x^\top P^*_{q_t}x &\le(-1+2r)(\gamma^{-1}Y_1)^2+\\ &\qquad(1+2r)(\gamma^{-1}Y_2)^2+(1+2r)X_1^2+(-1+2r)X_2^2\\ &\le 4(-1+2r)X_1^2+(1+2r)X_1^2+\tfrac{1}{4}(1+2r)X_2^2+(-1+2r)X_2^2\\ &=(-3+10r)X_1^2+\tfrac{1}{4}(-3+10r)X_2^2<0.\end{aligned}$$

The latter follows from the assumption that $r<\frac{3}{10}$. This proves that $\bar{A}_{w_t}^\top P^*_{q_{t+1}}\bar{A}_{w_t}-\gamma^2P^*_{q_t}\prec 0$, and thus it follows that $(q_t,w_t,q_{t+1})\in\Theta^*$, proving the claim at the beginning of the paragraph.

Finally, to conclude the proof of the theorem, it remains to show that, from Aut^* defined above, we can build an automaton $\mathsf{Aut}=(Q,\Sigma,\Theta)$ accepting every $\sigma\in\mathcal{S}$ such that SwS is p-dominant with Aut. This is done by splitting each transition $(q_1,w,q_2)\in\Theta^*$ of Aut^* into T sub-transitions (one per symbol of $w\in\Sigma^T$). More precisely, for each transition $\theta=(q_1,w,q_2)\in\Theta^*$, we add to $Q^*=\{1,\ldots,m\}$ the states $(\theta,1),\ldots,(\theta,T-1)$. This gives the set of states $Q=Q^*\cup(\Theta^*\times\{1,\ldots,T-1\})$. Because Q contains states from Q^* and states induced by the transitions in Θ^*, we introduce the following unifying notation: for $\theta=(q_1,w,q_2)\in\Theta^*$ and $k\in\{0,\ldots,T\}$, we let $\bar{q}(\theta,k)=q_1$ if $k=0$, $\bar{q}(\theta,k)=(\theta,k)$ if $1\le k\le T-1$, and $\bar{q}(\theta,k)=q_2$ if $k=T$, and for each $k\in\{0,\ldots,T-1\}$, we let $\bar{w}(\theta,k)=i_{k+1}$, where $w=(i_1,\ldots,i_T)$. Then, we define the set of transitions of Aut by $\Theta=\big\{(\bar{q}(\theta,k),\bar{w}(\theta,k),\bar{q}(\theta,k+1)) : \theta\in\Theta^*,\ 0\le k\le T-1\big\}$. By construction, it is clear that $\mathsf{Aut}=(Q,\Sigma,\Theta)$ accepts every $\sigma\in\mathcal{S}$.

It remains to show that that SwS is p-dominant with Aut, some set of rates $\{\gamma_\theta\}_{\theta\in\Theta}\subseteq\mathbb{R}_{>0}$ and some set of matrices $\{P_q\}_{q\in Q}\subseteq\mathbb{S}^{n\times n}_p$. The set of rates and the set of symmetric matrices are built as follows. Fix $\delta > 0$. For each transition $\theta = (q_1, w, q_2) \in \Theta^*$ in Aut^*, we let $P_{\bar{q}(\theta,0)} = P^*_{q_1}$ and $P_{\bar{q}(\theta,T)} = P^*_{q_2}$, and for $k = T-1, T-2, \ldots, 1$, we define the matrices $P_{\bar{q}(\theta,k)}$ recursively as follows:

$$P_{\bar{q}(\theta,k)} = (\gamma^*_\theta)^{-2/T} A^\top_{\bar{w}(\theta,k)} P_{\bar{q}(\theta,k+1)} A_{\bar{w}(\theta,k)} + \delta I.$$

By construction, we have that for all $\theta \in \Theta^*$ and $k \in \{1, \ldots, T-1\}$,

$$A^\top_{\bar{w}(\theta,k)} P_{\bar{q}(\theta,k+1)} A_{\bar{w}(\theta,k)} - (\gamma^*_\theta)^{2/T} P_{\bar{q}(\theta,k)} \prec 0. \tag{A.18}$$

Observe that $A^\top_{\bar{w}(\theta,0)} P_{\bar{q}(\theta,1)} A_{\bar{w}(\theta,0)} = (\gamma^*_\theta)^{2(1-1/T)} \bar{A}^\top_w P_{\bar{q}(\theta,T)} \bar{A}_w + \Delta$ where $\bar{A}_w = A_{\bar{w}(\theta,T-1)} \cdots A_{\bar{w}(\theta,0)}$, and $\Delta \in \mathbb{R}^{n\times n}$ satisfies $\|\Delta\| \in \mathcal{O}(\delta)$. Hence, by definition of γ^*_θ, it follows that (A.18) is also satisfied for $k = 0$, provided δ is small enough.

Summarizing, we have shown that the automaton Aut, together with the rates $\{\gamma_\theta\}_{\theta\in\Theta}$ defined by $\gamma_\theta = (\gamma^*_\theta)^{1/T}$ if $\theta = (\bar{q}(\theta^*, k), \bar{w}(\theta^*, k), \bar{q}(\theta^*, k+1))$ for some $\theta^* \in \Theta^*$ and $k \in \{0, \ldots, T-1\}$, and with the matrices $\{P_q\}_{q\in Q}$ defined as above, satisfy the dissipation inequalities (2.4). Hence, to show that SwS is p-dominant, it remains to show that $\{P_q\}_{q\in Q} \subseteq \mathbb{S}^{n\times n}_p$. By using (A.18) (which holds for all $k \in \{0, \ldots, T-1\}$) and Theorem A.40, we get that for every $\theta \in \Theta^*$,

$$\begin{aligned} p = \nu(P_{\bar{q}(\theta,T)}) \geq \nu_0(P_{\bar{q}(\theta,T-1)}) \geq \nu(P_{\bar{q}(\theta,T-1)}) \geq \cdots \geq \\ \nu_0(P_{\bar{q}(\theta,1)}) \geq \nu(P_{\bar{q}(\theta,1)}) \geq \nu_0(P_{\bar{q}(\theta,0)}) = p, \end{aligned}$$

whence for all $k \in \{0, \ldots, T\}$, $\nu(P_{\bar{q}(\theta,k)}) = \nu_0(P_{\bar{q}(\theta,k)}) = p$, concluding the proof that $2 \Rightarrow 1$. □

A.2.5 Proof of Theorem 2.15

Part 1: $1 \Rightarrow 2$

Assume that SwS satisfies Item 1. Then, the first assertion in Item 2 follows directly from Theorem 2.11. The second assertion in Item 2 follows from (A.16) and the fact that since Aut is cycle-stable with respect to $\{\gamma_\theta\}_{\theta\in\Theta}$ there is $M \geq 1$ such that $\gamma_{\theta_{t_0}} \cdots \gamma_{\theta_{t_1}} \leq M$ for every path $(\theta_t)_{t=0}^\infty$ in Aut and every $t_0, t_1 \in \mathbb{N}$, $t_1 \geq t_0$. Hence, it suffices to take $\rho = \sqrt{\mu}$ and $D = \sqrt{\epsilon^{-1} KM}$. □

Part 2: $2 \Rightarrow 1$

The proof is very similar to the proof of $2 \Rightarrow 1$ in Theorem 2.11. We just need to make the following modifications:

- We let $T \in \mathbb{N}$ be such that $C\mu^T \leq \frac{1}{4}$ and $D\rho^T < \frac{1}{2}$. The second constraint will imply that there is $\gamma \in (0,1)$ satisfying (A.17).
- We let $\theta \doteq (q_1, w, q_2) \in \Theta^*$ if and only if $\bar{A}_w^\intercal P_{q_2} \bar{A}_w - (\gamma_\theta^*)^2 P_{q_1} \prec 0$ for some $\gamma_\theta^* \in (0,1)$, where $\bar{A}_w = A_{i_T} \cdots A_{i_1}$ and $w = (i_1, \ldots, i_T)$.

The rest of the proof is exactly the same as the proof of 2 $\Rightarrow$ 1 in Theorem 2.11. Observe that since $\gamma_\theta^* < 1$ for all $\theta \in \Theta^*$, we have that $\gamma_\theta < 1$ for all $\theta \in \Theta$. Hence, the automaton $\mathsf{Aut} = (Q, \Sigma, \Theta)$ is cycle-stable with respect to $\{\gamma_\theta\}_{\theta\in\Theta}$. □

A.2.6 Proof of Theorem 2.17

Consider an automaton $\mathsf{Aut} = (Q, \Sigma, \Theta)$ satisfying Assumption 2.16. We say that $q \in Q$ is *recurrent* if there is a path $(\theta_t)_{t=0}^{T-1} \subseteq \Theta$ with length $T \in \mathbb{N}_{>0}$ from q to q, i.e., with $\mathsf{s}(\theta_0) = \mathsf{t}(\theta_{T-1}) = q$. Let $(\{P_q\}, \epsilon)$ be a feasible solution of (2.7b)–(2.7c) with $\epsilon > 0$.

We first show that for any recurrent state $q \in Q$ the inertia of P_q depends only on the automaton, the set of rates $\{\gamma_\theta\}_{\theta\in\Theta}$ and the matrices $\{A_i\}_{i\in\Sigma}$. To show this, fix a recurrent state $q \in Q$ and let $(\theta_t)_{t=0}^{T-1}$ be a path from q to itself. For every $t \in \{0, \ldots, T-1\}$, let $\bar{A}_t = A_{\mathsf{i}(\theta_t)} \cdots A_{\mathsf{i}(\theta_0)}$ and $\bar{\gamma}_t = \gamma_{\theta_{T-1}} \cdots \gamma_{\theta_t}$. Then, from (2.7b) and using that $P_q = P_{\mathsf{s}(\theta_0)} = P_{\mathsf{t}(\theta_{T-1})}$, we get that

$$\bar{A}_{T-1}^\intercal P_q \bar{A}_{T-1} \prec (\bar{\gamma}_{T-1})^2 \bar{A}_{T-2}^\intercal P_{\mathsf{t}(q_{T-2})} \bar{A}_{T-2} \prec (\bar{\gamma}_{T-2})^2 \bar{A}_{T-3}^\intercal P_{\mathsf{t}(q_{T-3})} \bar{A}_{T-3} \prec \ldots \prec (\bar{\gamma}_0)^2 P_q. \tag{A.19}$$

Hence, by Theorem A.42, we have that $P_q \in \mathbb{S}_{k_q}^{n\times n}$ where k_q is the number of eigenvalues of $\bar{A}_{T-1}$ with modulus $> \bar{\gamma}_0$. Because $\bar{A}_{T-1}$ and $\bar{\gamma}_0$ depend only on Aut, $\{\gamma_\theta\}_{\theta\in\Theta}$, and $\{A_i\}_{i\in\Sigma}$, and not on a particular solution $(\{P_q\}, \epsilon)$, and by hypothesis of Theorem 2.17, it follows that $\{P_q\}_{q\in Q} \subseteq \mathbb{S}_k^{n\times n}$.

Now, let $q \in Q$ be a non-recurrent state. By Assumption 2.16, there is a recurrent state q^- and a path $(\theta_t)_{t=0}^{T-1}$ from q^- to q (since any backward infinite path from q will eventually loop on itself). By the same argument as above, it holds that $\bar{A}^\intercal P_q \bar{A} - \bar{\gamma}^2 P_{q^-} \prec 0$, where $\bar{A} = A_{\mathsf{i}(\theta_{T-1})} \cdots A_{\mathsf{i}(\theta_0)}$ and $\bar{\gamma} = \gamma_{\theta_{T-1}} \cdots \gamma_{\theta_0}$. Hence, by Theorem A.40, it follows that $\nu(P_q) \geq \nu_0(P_{q^-}) = k$. By proceeding in a similar way (using a path from q to a recurrent state), we can show that $\nu_0(P_q) \leq k$. Hence, $\nu(P_q) = \nu_0(P_q) = k$, and thus $P_q \in \mathbb{S}_k^{n\times n}$, concluding the proof of the theorem. □

A.2.7 Proof of Proposition 2.19

Let $(\theta_t)_{t=0}^{T-1}$, $\bar{A}$ and $\bar{\gamma}$ be as in the proposition. By using the same argument as in (A.19), we get that $\bar{A}^\intercal P \bar{A} - \bar{\gamma}^2 P \prec 0$. Hence, by Theorem A.42, $\bar{A}$ has p

eigenvalues with modulus $|\lambda_i| > \bar{\gamma}$ and $n-p$ eigenvalues with modulus $|\lambda_i| < \bar{\gamma}$. This proves Item 1.

Now, to prove Item 2, let the columns of $H \in \mathbb{R}^{n\times p}$ be a basis of the eigenspace, denoted by F, associated to the p eigenvalues of $\bar{A}$ with modulus $|\lambda_i| > \bar{\gamma}$. Then, it holds that $\bar{A}H = H\bar{A}_p$ for some $\bar{A}_p \in \mathbb{R}^{p\times p}$ whose eigenvalues are equal to the eigenvalues of $\bar{A}$ satisfying $|\lambda_i| > \bar{\gamma}$. It follows from $\bar{A}H = H\bar{A}_p$ that $\bar{A}_p^\top H^\top P H \bar{A}_p - \bar{\gamma}^2 H^\top P H \prec 0$, and thus Theorem A.42 implies that $H^\top P H$ is negative definite. Hence, any $x \in F$ satisfies $x^\top P x \leq 0$, so that $F \subseteq \mathcal{K}(P)$. A similar reasoning shows that any x in the eigenspace associated to the $n-p$ eigenvalues of $\bar{A}$ with modulus $|\lambda_i| < \bar{\gamma}$ satisfies $x^\top P x \geq 0$. This concludes the proof of Item 2. □

A.2.8 Proof of Theorem 2.24

Let $(\{P_q\}_{q\in Q}, \{E_q\}_{q\in Q}, \{\delta_q\}_{q\in Q}, \eta, \epsilon)$ be a feasible solution of (2.11b)–(2.11d). We show that $\{P_q\}_{q\in Q}$ and ϵ satisfy (2.9b)–(2.9c). By assumption on $\{\delta_q\}_{q\in Q} \subseteq [0,1]$ and by the constraint (2.11d), it is clear that $\{P_q\}_{q\in Q}$ satisfies (2.9c). Hence, it remains to prove that $\{P_q\}_{q\in Q}$ and ϵ satisfy (2.9b). Therefore, fix $\theta \in \Theta$, and let $A = A^{\mathrm{c}}_{\mathsf{i}(\theta)} + \sum_{j=1}^{N_{\mathsf{i}(\theta)}} \alpha_j A^{\mathrm{h}}_{\mathsf{i}(\theta),j} + \Delta$ where $\alpha_1, \ldots, \alpha_{N_{\mathsf{i}(\theta)}} \in \mathbb{R}_{\geq 0}$ satisfy $\sum_{j=1}^{N_{\mathsf{i}(\theta)}} \alpha_j = 1$ and $\Delta \in r_{\mathsf{i}(\theta)}\mathbb{B}$. We show that $\{P_q\}_{q\in Q}$ and ϵ satisfy (2.9b) with this θ and this $A \in \mathcal{A}_i$. Indeed, from (2.11b), it follows that

$$\begin{aligned}
A^\top P_{\mathsf{t}(\theta)} A - \gamma_\theta^2 P_{\mathsf{s}(\theta)} &= {A^{\mathrm{c}}_{\mathsf{i}(\theta)}}^\top P_{\mathsf{t}(\theta)} A^{\mathrm{c}}_{\mathsf{i}(\theta)} + \textstyle\sum_j \alpha_j ({A^{\mathrm{c}}_{\mathsf{i}(\theta)}}^\top P_{\mathsf{t}(\theta)} A^{\mathrm{h}}_{\mathsf{i}(\theta),j} + {A^{\mathrm{h}}_{\mathsf{i}(\theta),j}}^\top P_{\mathsf{t}(\theta)} A^{\mathrm{c}}_{\mathsf{i}(\theta)}) \\
&\quad + \textstyle\sum_j \alpha_j ((A^{\mathrm{c}}_{\mathsf{i}(\theta)} + A^{\mathrm{h}}_{\mathsf{i}(\theta),j})^\top P_{\mathsf{t}(\theta)} \Delta + \Delta^\top P_{\mathsf{t}(\theta)} (A^{\mathrm{c}}_{\mathsf{i}(\theta)} + A^{\mathrm{h}}_{\mathsf{i}(\theta),j})) \\
&\quad + (\textstyle\sum_j \alpha_j A^{\mathrm{h}}_{\mathsf{i}(\theta),j})^\top P_{\mathsf{t}(\theta)} (\textstyle\sum_j \alpha_j A^{\mathrm{h}}_{\mathsf{i}(\theta),j}) + \Delta^\top P_{\mathsf{t}(\theta)} \Delta - \gamma_\theta^2 P_{\mathsf{s}(\theta)}
\end{aligned}$$

(then, using $\delta_{\mathsf{t}(\theta)} \geq \|P_{\mathsf{t}(\theta)}\|$ and $\eta \geq r_{\mathsf{i}(\theta)}^2 \delta_{\mathsf{t}(\theta)} + 2r_{\mathsf{i}(\theta)} \|P_{\mathsf{t}(\theta)}(A^{\mathrm{c}}_{\mathsf{i}(\theta)} + A^{\mathrm{h}}_{\mathsf{i}(\theta),j})\|$)

$$\begin{aligned}
&\preceq {A^{\mathrm{c}}_{\mathsf{i}(\theta)}}^\top P_{\mathsf{t}(\theta)} A^{\mathrm{c}}_{\mathsf{i}(\theta)} + \textstyle\sum_j \alpha_j ({A^{\mathrm{c}}_{\mathsf{i}(\theta)}}^\top P_{\mathsf{t}(\theta)} A^{\mathrm{h}}_{\mathsf{i}(\theta),j} + {A^{\mathrm{h}}_{\mathsf{i}(\theta),j}}^\top P_{\mathsf{t}(\theta)} A^{\mathrm{c}}_{\mathsf{i}(\theta)}) \\
&\quad + (\textstyle\sum_j \alpha_j A^{\mathrm{h}}_{\mathsf{i}(\theta),j})^\top P_{\mathsf{t}(\theta)} (\textstyle\sum_j \alpha_j A^{\mathrm{h}}_{\mathsf{i}(\theta),j}) + \eta I - \gamma_\theta^2 P_{\mathsf{s}(\theta)}
\end{aligned}$$

(then, using $E_{\mathsf{t}(\theta)} \succeq P_{\mathsf{t}(\theta)}$ and $E_{\mathsf{t}(\theta)} \succeq 0$)

$$\begin{aligned}
&\preceq {A^{\mathrm{c}}_{\mathsf{i}(\theta)}}^\top P_{\mathsf{t}(\theta)} A^{\mathrm{c}}_{\mathsf{i}(\theta)} + \textstyle\sum_j \alpha_j ({A^{\mathrm{c}}_{\mathsf{i}(\theta)}}^\top P_{\mathsf{t}(\theta)} A^{\mathrm{h}}_{\mathsf{i}(\theta),j} + {A^{\mathrm{h}}_{\mathsf{i}(\theta),j}}^\top P_{\mathsf{t}(\theta)} A^{\mathrm{c}}_{\mathsf{i}(\theta)}) \\
&\quad + \textstyle\sum_j \alpha_j {A^{\mathrm{h}}_{\mathsf{i}(\theta),j}}^\top E_{\mathsf{t}(\theta)} A^{\mathrm{h}}_{\mathsf{i}(\theta),j} + \eta I - \gamma_\theta^2 P_{\mathsf{s}(\theta)} \preceq -\epsilon I,
\end{aligned}$$

where the last step follows from (2.11b). This concludes the proof of the theorem. □

A.2.9 Proof of Proposition 2.27

Part 1: $1 \Rightarrow 2$

Assume that Sys is p-dominant on Λ with $\mathsf{Aut} = (Q, \Sigma, \Theta)$, $\{\mathcal{A}_i\}_{i\in\Sigma}$ and $\{\gamma_\theta\}_{\theta\in\Theta}$ satisfying $\gamma_\theta = 1$ for all $\theta \in \Theta$. Note that Aut is cycle-stable with respect to $\{\gamma_\theta\}_{\theta\in\Theta}$. Also note that, for any $x \in \Lambda$ and $T \in \mathbb{N}$, $\frac{\partial\chi}{\partial x}(T, x) = \frac{\partial f}{\partial x}(\chi(T-1, x)) \cdots \frac{\partial f}{\partial x}(\chi(0, x))$, and by definition of Aut and $\{\mathcal{A}_i\}_{i\in\Sigma}$, for any $x \in \Lambda$, there is a path $(\theta_t)_{t=0}^{\infty}$ in Aut, such that for all $t \in \mathbb{N}$, $\frac{\partial f}{\partial x}(\chi(t, x)) \in \mathcal{A}_{\mathsf{i}(\theta_t)}$. Hence, by exactly the same argument as in the proof of $1 \Rightarrow 2$ in Theorem 2.15, it follows that there is $D_1 \geq 1$ and $\rho_1 \in (0, 1)$, and for each $x \in \Lambda$, there is a subspace $E^{\mathrm{s}}(x) \subseteq \mathbb{R}^n$ of dimension $n - p$ such that for all $v \in E^{\mathrm{s}}(x)$ and $t \in \mathbb{N}$, $\|\frac{\partial\chi}{\partial x}(t, x)v\| \leq \|v\| D_1 \rho_1^t$. Since Sys is invertible, and Λ is invariant, we obtain, with the same argument applied on the inverted system $(\mathbb{R}^n, f^{-1})$, that there is $D_2 \geq 1$ and $\rho_2 \in (0, 1)$, and for each $x \in \Lambda$, there is a subspace $E^{\mathrm{u}}(x) \subseteq \mathbb{R}^n$ of dimension p such that for all $v \in E^{\mathrm{u}}(x)$ and $t \in \mathbb{N}$, $\|\frac{\partial\chi}{\partial x}(-t, x)v\| \leq \|v\| D_2 \rho_2^t$.

To prove that Λ is hyperbolic for Sys, it remains to show that for all $x \in \Lambda$, $t \in \mathbb{N}$ and $\diamond \in \{\mathrm{s}, \mathrm{u}\}$, $\frac{\partial\chi}{\partial x}(t, x)E^\diamond(x) = E^\diamond(\chi(t, x))$. We proceed by contradiction. Therefore, fix $t \in \mathbb{N}$ and $x \in \Lambda$, and assume that $\frac{\partial\chi}{\partial x}(t, x)E^{\mathrm{s}}(x) \neq E^{\mathrm{s}}(\chi(t, x))$, so that $\dim\left(\frac{\partial\chi}{\partial x}(t, x)E^{\mathrm{s}}(x) + E^{\mathrm{s}}(\chi(t, x))\right) > n - p$. Then, fix $t' \in \mathbb{N}$ and let v be a nonzero vector in $\frac{\partial\chi}{\partial x}(t', \chi(t, x))(\frac{\partial\chi}{\partial x}(t, x)E^{\mathrm{s}}(x) + E^{\mathrm{s}}(\chi(t, x))) \cap E^{\mathrm{u}}(\chi(t' + t, x))$. On the one hand, it holds that $\|\frac{\partial\chi}{\partial x}(-t', \chi(t + t', x))v\| \leq \|v\| D_2 \rho_2^{t'}$, and on the other hand, it holds that $\|v\| \leq \|\frac{\partial\chi}{\partial x}(-t', \chi(t+t', x))v\| \tilde{D} D_1 \rho_1^{t'}$, for some $\tilde{D} \geq 1$ is independent from t'. The two inequalities are not compatible when t' is sufficiently large. Since t' was arbitrary, we obtain a contradiction. This shows that $\frac{\partial\chi}{\partial x}(t, x)E^{\mathrm{s}}(x) = E^{\mathrm{s}}(\chi(t, x))$. Using the exact same argument in backward time, we obtain that $\frac{\partial\chi}{\partial x}(t, x)E^{\mathrm{u}}(x) = E^{\mathrm{u}}(\chi(t, x))$. Since x and t were arbitrary, this proves the claim at the beginning of the paragraph, which concludes the proof that Λ is hyperbolic for Sys. □

Part 2: $2 \Rightarrow 1$

Assume that Λ is a bounded, connected, hyperbolic invariant set for Sys, and let $E^{\mathrm{s}} : \Lambda \rightrightarrows \mathbb{R}^n$ and $E^{\mathrm{u}} : \Lambda \rightrightarrows \mathbb{R}^n$ be as in Definition 1.31 for the hyperbolicity of Λ. By Remark 1.5, it holds that there is $p \in \{0, \ldots, n\}$ such that for all $x \in \Lambda$, $\dim E^{\mathrm{s}}(x) = n - p$ and $\dim E^{\mathrm{u}}(x) = p$. The rest of the proof is exactly the same as the proof of $2 \Rightarrow 1$ in Theorem 2.11. We just want to make the following observations regarding the adaptation of this proof:

- Lemma A.44 holds trivially since Λ is bounded and Sys is invertible, so that $\inf_{x\in\Lambda} \left\|\frac{\partial f^{-1}}{\partial x}(x)\right\| < \infty$;

- By the hypothesis of hyperbolicity, we can choose $\gamma_\theta = 1$ for all $\theta \in \Theta$.

This concludes the proof that $2 \Rightarrow 1$. □

A.2.10 Proof of Proposition 2.28

The proof of the upper bound relies on the following result.

Theorem A.46 (Matveev and Pogromsky, 2019, Theorem 11)**.** *Consider a dynamical system* $\mathsf{Sys} = (\mathbb{R}^n, f)$ *and let* $\Lambda \subseteq \mathbb{R}^n$ *be a bounded forward invariant set for* Sys. *Let* $h_{\mathrm{top}}(\mathsf{Sys}, \Lambda)$ *be the topological entropy of* Sys *restricted to* Λ *with cost function* $(x_1, x_2) \mapsto \|x_1 - x_2\|$. *It holds that*

$$h_{\mathrm{top}}(\mathsf{Sys}, \Lambda) \leq \limsup_{T\to\infty} \frac{1}{T} \left(\sup_{x\in\Lambda} \sum\nolimits_{i=1}^{n} \max\left\{0, \log_2 \rho_i\left(\tfrac{\partial\chi}{\partial x}(T, x)\right)\right\}\right),$$

where $\rho_1(A), \ldots, \rho_n(A)$ *denote the singular values of* $A \in \mathbb{R}^{n\times n}$.

Assume that Sys is p-dominant on Λ with automaton $\mathsf{Aut} = (Q, \Sigma, \Theta)$ and set of rates $\{\gamma_\theta\}_{\theta\in\Theta}$, and let $\hat{\gamma}_{\max} = \max\{\, (\gamma_{\theta_0} \cdots \gamma_{\theta_{T-1}})^{1/T} : (\theta_t)_{t=0}^{T-1} \text{ is a cycle in } \mathsf{Aut}\,\}$. We show that, for $T \in \mathbb{N}$ sufficiently large, $\frac{\partial\chi}{\partial x}(T, x)$ has $n - p$ singular values smaller than $(\hat{\gamma}_{\max})^T$. Indeed, by the same argument as in the proof of $1 \Rightarrow 2$ in Proposition 2.27, it follows that there is $C \geq 1$ and $\mu \in (0, 1)$, and for each $x \in \Lambda$, there is a subspace $E^{\mathrm{s}}(x) \subseteq \mathbb{R}^n$ of dimension $n - p$ such that for all $v \in E^{\mathrm{s}}(x)$ and $t \in \mathbb{Z}_{\geq 0}$, $\|\frac{\partial\chi}{\partial x}(t, x)v\| \leq \|v\| D\mu^t (\hat{\gamma}_{\max})^t$. Hence, if we take $T \in \mathbb{N}$ such $D\mu^T < 1$, it holds that for any $x \in \Lambda$ and $v \in E^{\mathrm{s}}(x)$, $\|\frac{\partial\chi}{\partial x}(T, x)v\| \leq \|v\|(\hat{\gamma}_{\max})^T$. Then, by Theorem A.41 applied on the symmetric matrix $\frac{\partial\chi}{\partial x}(T, x)^\top \frac{\partial\chi}{\partial x}(T, x) - (\hat{\gamma}_{\max})^{2T} I$, it follows that $\frac{\partial\chi}{\partial x}(T, x)^\top \frac{\partial\chi}{\partial x}(T, x)$ has $n - p$ eigenvalues smaller than $(\hat{\gamma}_{\max})^{2T}$, so that $\frac{\partial\chi}{\partial x}(T, x)$ has $n - p$ singular values smaller than $(\hat{\gamma}_{\max})^T$, proving the claim at the beginning of the paragraph.

Now, let p_k and $\hat{\gamma}_{\max}^{(k)}$, for each $k \in \{1, \ldots, n\}$, be as in the proposition, and let $T \in \mathbb{N}$ be large enough. Then, it follows from the above that for any $x \in \Lambda$ and $k \in \{1, \ldots, n\}$, $\frac{\partial\chi}{\partial x}(T, x)$ has $n - p_k$ singular values smaller than $(\hat{\gamma}_{\max}^{(k)})^T$. Hence, for every $x \in \Lambda$, the singular values of $\frac{\partial\chi}{\partial x}(T, x)$ can be indexed in such a way that for each $k \in \{1, \ldots, n\}$, the k^{th} singular value of $\frac{\partial\chi}{\partial x}(T, x)$ is smaller than $(\hat{\gamma}_{\max}^{(k)})^T$. Since T and x were arbitrary (provided T is sufficiently large), we obtain the desired upper bound on $h_{\mathrm{top}}(\mathsf{Sys}, \Lambda)$ by using Theorem A.46.

The proof of the lower bound is along the same lines, and thus omitted. This concludes the proof of the theorem. □

A.3 Proofs of Chapter 3

A.3.1 Proof of Lemma 3.6

Let $\Lambda = \mathbb{B}_{\mathsf{Sys},T}(a, c\epsilon)$, and let $F \subseteq \Lambda$ be a set with maximal cardinality such that $\{\chi(\cdot, x)|_{[0,T)\cap\mathbb{T}}\}_{x\in F}$ is (ϵ, T)-separated for Sys starting from Λ. We derive an upper bound on the cardinality of F, as follows. Since $\{\chi(\cdot, x)|_{[0,T)\cap\mathbb{T}}\}_{x\in F}$ is (ϵ, T)-separated for Sys, it holds that for any distinct $x_1, x_2 \in F$, $\mathbb{B}_{\mathsf{Sys},T}(x_1, \epsilon/2)\cap \mathbb{B}_{\mathsf{Sys},T}(x_2, \epsilon/2) = \emptyset$. It follows that $\mathrm{vol}(\bigcup_{x\in F} \mathbb{B}_{\mathsf{Sys},T}(x, \epsilon/2)) = |F| \,\mathrm{vol}(\mathbb{B}_{\mathsf{Sys},T}(x, \epsilon/2))$, where "vol" stands for the Lebesgue measure (or "volume"); note that $\mathbb{B}_{\mathsf{Sys},T}(b, r)$ is Lebesgue measurable for any $b \in \mathbb{R}^n$ and $r \geq 0$, since it is the unit ball of a norm and thus a convex set in $\mathbb{R}^n$ (see, e.g., Lang, 1986). On the other hand, since $F \subseteq \Lambda$, it holds that $\bigcup_{x\in F} \mathbb{B}_{\mathsf{Sys},T}(x, \epsilon/2)) \subseteq \mathbb{B}_{\mathsf{Sys},T}(a, (c+1/2)\epsilon)$. Putting things together, we get that

$$|F| \leq \frac{\mathrm{vol}(\mathbb{B}_{\mathsf{Sys},T}(a, (c + 1/2)\epsilon))}{\mathrm{vol}(\mathbb{B}_{\mathsf{Sys},T}(0, \epsilon/2))} = (2c + 1)^n.$$

The latter comes from $\mathrm{vol}(\mathbb{B}_{\mathsf{Sys},T}(b, r)) = r^n \mathrm{vol}(\mathbb{B}_{\mathsf{Sys},T}(0, 1)) \neq 0$ for any $b \in \mathbb{R}^n$ and $r > 0$, which follows from $\mathbb{B}_{\mathsf{Sys},T}(b, r) = b + r\mathbb{B}_{\mathsf{Sys},T}(0, 1)$, so that

$$\begin{aligned}
\mathrm{vol}(\mathbb{B}_{\mathsf{Sys},T}(b, r)) &= \int_{\mathbb{B}_{\mathsf{Sys},T}(b,r)} \mathrm{d}\lambda(x) = \int_{\mathbb{R}^n} \mathbf{1}_{\mathbb{B}_{\mathsf{Sys},T}(b,r)}(x)\, \mathrm{d}\lambda(x) \\
&= \int_{\mathbb{R}^n} \mathbf{1}_{\mathbb{B}_{\mathsf{Sys},T}(0,1)}(r^{-1}(x - b))\, \mathrm{d}\lambda(x) \\
&= \int_{\mathbb{R}^n} \mathbf{1}_{\mathbb{B}_{\mathsf{Sys},T}(0,1)}(y) r^n\, \mathrm{d}\lambda(y) = r^n \mathrm{vol}(\mathbb{B}_{\mathsf{Sys},T}(0, 1)) \neq 0,
\end{aligned}$$

where λ is the Lebesgue measure, $\mathbf{1}$ is the indicator function, and the penultimate step follows from the change of variable formula for the Lebesgue integral (see, e.g., Teschl, 2021, Theorem 2.17).

The proof is then complete by using Lemma 3.5, which implies that F is an (ϵ, T)-cover of Λ. □

A.3.2 Proof of Lemma 3.7

We will need the following result.

Lemma A.47. *Consider a LTV system* $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$ *and a set* $\Lambda \subseteq \mathbb{R}^n$*. Let* $\epsilon > 0$ *and* $T \in \mathbb{T}_{>0}$*. Let* E *be a minimal* (ϵ, T)*-cover of* Λ*. Then, there exists a subset* $F \subseteq E$*, with cardinality* $|F| \geq 11^{-n}|E|$*, such that for any distinct* $x_1, x_2 \in F$*,* $\|x_1 - x_2\|_{\mathsf{Sys},T} > 4\epsilon$*.*

Proof of Lemma A.47. Fix $x \in E$, and let $E_x = \mathbb{B}_{\mathsf{Sys},T}(x, 4\epsilon)\cap E$. We first show that $|E_x| \leq 11^n$. Indeed, it holds that $\bigcup_{x'\in E_x} \mathbb{B}_{\mathsf{Sys},T}(x', \epsilon) \subseteq \mathbb{B}_{\mathsf{Sys},T}(x, 5\epsilon)$, and we have seen in Lemma 3.6 that there exists an (ϵ, T)-cover E^* of $\mathbb{B}_{\mathsf{Sys},T}(x, 5\epsilon)$ with $|E^*| \leq 11^n$. Thus, from the minimality of E, it follows that $|E_x| \leq 11^n$.

Now, using the above, we build the set F inductively as follows. Let $F_0 = \emptyset$ and $G_0 = E$. Then, for $i = 0, 1, 2, \ldots$ and while $G_i \neq \emptyset$, pick $x_i \in G_i$ and let $F_{i+1} = F_i \cup \{x_i\}$ and $G_{i+1} = G_i \setminus E_{x_i}$, where E_x is defined as above. Let k be the first integer such that $G_k = \emptyset$ and define $F = F_k$. Because $|E_{x_i}| \leq 11^n$ for each $0 \leq i \leq k-1$, we have that $k \geq 11^{-n}|E|$. This concludes the proof of the lemma. □

We proceed with the proof of Lemma 3.7. Note that since E_1 is a minimal (ϵ, T_1)-cover of Λ, it holds that for all $x \in E_1$, $\mathbb{B}_{\mathsf{Sys},T_1}(x, \epsilon) \cap \Lambda \neq \emptyset$, which implies that $\mathbb{B}_{\mathsf{Sys},T_1}(x, \epsilon) \subseteq \Lambda + \mathbb{B}_{\mathsf{Sys},T_1}(0, 2\epsilon)$. Using Lemma A.47, let $F_1 \subseteq E_1$ be such that $|F_1| \geq 11^{-n}|E_1|$ and for every distinct $x_1, x_2 \in F_1$, $\|x_1 - x_2\|_{\mathsf{Sys},T_1} > 4\epsilon$. Then, for each $x \in F_1$, let $E_x^* \subseteq E_2$ be an (ϵ, T_2)-cover of $\mathbb{B}_{\mathsf{Sys},T_1}(x, \epsilon)$ and without loss of generality, assume that for all $x' \in E_x^*$, $\mathbb{B}_{\mathsf{Sys},T_2}(x', \epsilon) \cap \mathbb{B}_{\mathsf{Sys},T_1}(x, \epsilon) \neq \emptyset$.

By hypothesis on F_1 and since $T_2 \geq T_1$, it holds that for any $x \in \mathbb{R}^n$, the ball $\mathbb{B}_{\mathsf{Sys},T_2}(x, \epsilon)$ cannot intersect simultaneously $\mathbb{B}_{\mathsf{Sys},T_1}(x_1, \epsilon)$ and $\mathbb{B}_{\mathsf{Sys},T_1}(x_2, \epsilon)$ if $x_1, x_2 \in F_1$ and $x_1 \neq x_2$. Thus, the subsets $\{E_x^*\}_{x \in F_1}$ defined above are pairwise disjoint. This implies that $\sum_{x \in F_1} |E_x^*| \leq |E_2|$, which in turn implies that $\min_{x \in F_1} |E_x^*| \leq |E_2|/|F_1| \leq 11^n |E_2|/|E_1|$. Now, it is not difficult to see that if E_x^* is an (ϵ, T_2)-cover of $\mathbb{B}_{\mathsf{Sys},T_1}(x, \epsilon)$, then $E_x^* - x$ is an (ϵ, T_2)-cover of $\mathbb{B}_{\mathsf{Sys},T_1}(0, \epsilon)$. This concludes the proof of the lemma. □

A.3.3 Proof of Lemma 3.8

Let $T_* \in (T_*, \infty)$ be such that $\|\mathring{\chi}(T', T)\| \leq c \doteq \frac{3}{2}$ for all $T' \in [T, T_*]$, where $\mathring{\chi}$ is the fundamental matrix solution of Sys. By definition of c, it holds that for every $x \in \mathbb{R}^n$, $\|x\|_{\mathsf{Sys},T_*} \leq c\|x\|_{\mathsf{Sys},T}$, so that $\mathbb{B}_{\mathsf{Sys},T}(0, \epsilon) \subseteq \mathbb{B}_{\mathsf{Sys},T_*}(0, c\epsilon)$. Thus, by Lemma 3.6, there is an (ϵ, T_*)-cover E' of $\mathbb{B}_{\mathsf{Sys},T}(0, \epsilon)$ with cardinality $|E'| \leq (2c+1)^n$. Finally, let E be a minimal (ϵ, T)-cover of Λ, and define $E_* = E + E'$. Clearly, E_* is an (ϵ, T_*)-cover of Λ and its cardinality satisfies $|E_*| \leq (2c+1)^n |E| = 4^n s_{\mathrm{cov}}(\epsilon, T; \Lambda)$. This concludes the proof of the lemma. □

A.3.4 Proof of Theorem 3.11

The proof of the upper bound relies on the following result.

Theorem A.48 (Vicinansa and Liberzon, 2019, Theorem 3.1)**.** *Consider a LTV system* $\mathsf{Sys} = (\mathbb{R}^n, \hat{A})$*. Let* $\{v_i\}_{i=1}^n \subseteq \mathbb{R}^n$ *be a basis for* $\mathbb{R}^n$*, and for each* $i \in \{1, \ldots, n\}$*, let* λ_i *be the* Lyapunov exponent *(in base 2) of* v_i *with respect to* Sys*, defined by* $\lambda_i = \limsup_{T \to \infty} \frac{1}{T} \log_2 \|\chi(T, 0, v_i)\|$*. Then, it holds that* $h_{\mathrm{top}}(\mathsf{Sys}) \leq \sum_{i=1}^n \max\{\lambda_i, 0\}$*.*

We proceed with the proof of the upper bound in Theorem 3.11. Since the system is p-dominant, by Theorem 2.11, there is a dominated p-splitting $(E^{\mathrm{s}}_\sigma, E^{\mathrm{u}}_\sigma)$ associated to σ. Let $\{v_i\}_{i=1}^n \subseteq \mathbb{R}^n$ be a basis of $\mathbb{R}^n$ satisfying $v_i \in E^{\mathrm{u}}_\sigma$ for all $i \in \{1,\ldots,p\}$ and $v_i \in E^{\mathrm{s}}_\sigma$ for all $i \in \{p+1,\ldots,n\}$. By (A.16), it holds that for all $i \in \{p+1,\ldots,n\}$, $\lambda_i \leq \limsup_{T\to\infty} \frac{1}{T}\log_2(\gamma_{\theta_0}\cdots\gamma_{\theta_{T-1}})$. On the other hand, there is $C \geq 0$ such that, for any $v \in \mathbb{R}^n$ and $T \in \mathbb{N}$, it holds that $\|\chi(T,0,v)\| \leq C\prod_{i\in\Sigma}\|A_i\|_*^{T\rho_i(T)}\|v\|$. It follows that for each $i \in \{1,\ldots,n\}$, $\lambda_i \leq \limsup_{T\to\infty}\sum_{i\in\Sigma}\rho_i(T)\log_2\|A_i\|_*$. This proves the upper bound.

The proof of the lower bound is along the same lines, and thus omitted. This concludes the proof of the theorem. □

A.3.5 Proof of Theorem 3.23

Part 1: $h_{\text{wc-top}}(\mathsf{SwS}) \leq \log_2(e)\hat{\varrho}(\mathsf{SwS}^\wedge)$

In the following, we let $\mathbb{B}$ be the centered unit Euclidean ball in $\mathbb{R}^n$. We will need the following lemma.

Lemma A.49. *Consider a switched linear system* $\mathsf{SwS} \sim (\mathbb{R}^n, \{A_i\}_{i\in\Sigma})$ *under arbitrary switching, and let* $R > \log_2(e)\hat{\varrho}(\mathsf{SwS})$*. There is* $T \in \mathbb{T}_{>0}$ *and for every* $\sigma \in \mathcal{S}$*, there is* $E \subseteq \mathbb{R}^n$ *such that (i)* $|E| \leq 2^{RT}$ *and (ii) for every* $x \in \mathbb{B}$*, there is* $\hat{x} \in E$ *satisfying* $\|\chi(T,0,x,\sigma) - \hat{x}\| \leq 1$*.*

Proof. The proof relies on the finite-points quantizer described in Figure 3.5 (in Subsection 3.3.3). Indeed, by Lemma 3.31, there is $T \in \mathbb{T}_{>0}$ such that for all $\sigma \in \mathcal{S}$, the quantizer $Q_{1,\mathring{\chi}(T,0,\sigma)} : \mathbb{R}^n \to \Xi_{1,\mathring{\chi}(T,0,\sigma)}$ (see Definition 3.29) satisfies that $|\Xi_{1,\mathring{\chi}(T,0,\sigma)}| \leq 2^{RT}$. Moreover, by Item 1 in Lemma 3.30, it holds that for all $\sigma \in \mathcal{S}$ and $T \in \mathbb{T}_{\geq 0}$, the set $\Xi_{1,\mathring{\chi}(T,0,\sigma)}$ satisfies that for every $x \in \mathbb{B}$, there is $\hat{x} \in \Xi_{1,\mathring{\chi}(T,0,\sigma)}$ such that $\|\chi(T,0,x,\sigma) - \hat{x}\| \leq 1$. This concludes the proof of the lemma. □

To prove that $h_{\text{wc-top}}(\mathsf{SwS}) \leq \log_2(e)\hat{\varrho}(\mathsf{SwS}^\wedge)$, let $\sigma \in \mathcal{S}$ and $X_0 \subseteq \mathbb{R}^n$ be a bounded set. We will show that $h_{\text{top}}(\mathsf{SwS};\sigma,X_0) \leq \log_2(e)\hat{\varrho}(\mathsf{SwS}^\wedge)$, where $h_{\text{top}}(\mathsf{SwS};\sigma,X_0)$ is the topological entropy of SwS with switching signal σ and starting from X_0. Therefore, fix $\epsilon > 0$ and $R > \log_2(e)\hat{\varrho}(\mathsf{SwS}^\wedge)$. Using Lemma A.49, let $T \in \mathbb{T}_{>0}$ be such that for each $k \in \mathbb{N}$, there is a set $E_k \subseteq \mathbb{R}^n$ such that (i) $|E_k| \leq 2^{RT}$ and (ii) for every $x \in \mathbb{B}$, there is $\hat{x} \in E_k$ satisfying $\|\chi((k+1)T, kT, x, \sigma) - \hat{x}\| \leq 1$. For each $k \in \mathbb{N}$, let the elements of E_k be indexed as follows: $E_k = \{\hat{x}_{k,j}\}_{j=1}^{m_k}$, where $m_k \leq 2^{RT}$. Also, let $\alpha > 0$ be such that for all $k \in \mathbb{N}$ and $t \in [kT,(k+1)T) \cap \mathbb{T}$, $\|\mathring{\chi}(t,kT,\sigma)\| \leq \epsilon/\alpha$, and let $E_{-1} \subseteq \mathbb{R}^n$ be a finite set such that for every $x \in X_0$, there is $\hat{x} \in E_{-1}$ such that $\|x/\alpha - \hat{x}\| \leq 1$ (which is always possible by boundedness of X_0). Let the elements of E_{-1} be indexed as follows: $E_{-1} = \{\hat{x}_{-1,j}\}_{j=1}^{m_{-1}}$.

We will use the above defined sets $\{E_k\}_{k=-1}^{\infty}$ to construct spanning sets for SwS with switching signal σ and starting from X_0. Therefore, fix $k \in \mathbb{N}$ and define the following set of functions from $[0, kT) \cap \mathbb{T}$ to $\mathbb{R}^n$:

$$\mathcal{E} = \{\, \hat{\xi}_{j_{-1},\ldots,j_{k-2}} \,\}_{j_\ell \in \{1,\ldots,m_\ell\},\, \ell \in \{-1,\ldots,k-2\}},$$

where for every $(j_{-1}, \ldots, j_{k-2}) \in \{1, \ldots, m_{-1}\} \times \cdots \times \{1, \ldots, m_{k-2}\}$, the function $\hat{\xi}_{j_{-1},\ldots,j_{k-2}} : [0, kT) \cap \mathbb{T} \to \mathbb{R}^n$ is defined by $\hat{\xi}_{j_{-1},\ldots,j_{k-2}}(t) = \sum_{\ell=-1}^{k-2} \alpha\, \chi(t, (\ell+1)T, \hat{x}_{\ell,j_\ell}, \sigma)$, with the convention that for any $t_0, t_1 \in \mathbb{T}$ and $x \in \mathbb{R}^n$, $\chi(t_1, t_0, x, \sigma) = 0$ if $t_1 < t_0$. We show that $\mathcal{E}$ is (ϵ, kT)-spanning for SwS with switching signal σ and starting from X_0.

To do this this, let $x \in X_0$, and define the indices $j_{-1}, \ldots, j_{k-2}$ inductively as follows. For each $\ell = -1, \ldots, k-2$, let $j_\ell \in \{1, \ldots, m_\ell\}$ be such that

$$\left\|\chi((\ell+1)T, 0, x, \sigma) - \textstyle\sum_{\ell'=-1}^{\ell-1} \alpha\, \chi((\ell+1)T, (\ell'+1)T, \hat{x}_{\ell',j_{\ell'}}, \sigma) - \alpha\, \hat{x}_{\ell,j_\ell}\right\| \leq \alpha. \tag{A.20}$$

We show, by induction on ℓ, that for each $\ell \in \{-1, \ldots, k-2\}$, there is an index $j_\ell \in \{1, \ldots, m_\ell\}$ satisfying (A.20). Indeed, this is trivially true for $\ell = -1$, by definition of E_{-1}. Now, assume that it holds for some $\ell \in \{-1, \ldots, k-3\}$. Then, $\chi((\ell+1)T, 0, x, \sigma) - \sum_{\ell'=-1}^{\ell} \alpha\, \chi((\ell+1)T, (\ell'+1)T, \hat{x}_{\ell',j_{\ell'}}, \sigma) \in \alpha\mathbb{B}$, so that, by definition of $E_{\ell+1}$ there is $j_{\ell+1} \in \{1, \ldots, m_{\ell+1}\}$ such that (A.20) is satisfied with ℓ replaced by $\ell+1$. This shows the induction step, concluding the proof by induction that for each $\ell \in \{-1, \ldots, k-2\}$, there is an index $j_\ell \in \{1, \ldots, m_\ell\}$ satisfying (A.20). Finally, we show that $\hat{\xi}_{j_{-1},\ldots,j_{k-2}}$ satisfies that for all $t \in [0, kT) \cap \mathbb{T}$, $\|\chi(t, 0, x, \sigma) - \hat{\xi}_{j_{-1},\ldots,j_{k-2}}(t)\| \leq \epsilon$. Indeed, let $\ell \in \{0, \ldots, k-1\}$ and $t \in [\ell T, (\ell+1)T) \cap \mathbb{T}$. Then, by (A.20), it holds that $\|\chi(\ell T, 0, x, \sigma) - \hat{\xi}_{j_{-1},\ldots,j_{k-2}}(\ell T)\| \leq \alpha$, and thus, by definition of α, it holds that $\|\chi(t, 0, x, \sigma) - \hat{\xi}_{j_{-1},\ldots,j_{k-2}}(t)\| \leq \epsilon$. Since ℓ and t were arbitrary, this holds for all $t \in [0, kT) \cap \mathbb{T}$. Since x was arbitrary, this shows that $\mathcal{E}$ is (ϵ, kT)-spanning for SwS with switching signal σ and starting from X_0.

The cardinality of $\mathcal{E}$ satisfies $|\mathcal{E}| \leq |E_{-1}| 2^{(k-1)RT}$. Since k was arbitrary, it follows that for every $k \in \mathbb{N}$, $s_{\text{span}}(\epsilon, kT; X_0) \leq |E_{-1}| 2^{(k-1)RT}$, and thus for all $T' \in \mathbb{T}_{\geq 0}$, $s_{\text{span}}(\epsilon, T'; X_0) \leq |E_{-1}| 2^{T'R}$. Hence, $\limsup_{T' \to \infty} \frac{1}{T'} \log_2 s_{\text{span}}(\epsilon, T'; X_0) \leq R$. Since ϵ was arbitrary, this holds for all $\epsilon > 0$. Thus, by definition of $h_{\text{top}}(\mathsf{SwS}; \sigma, X_0)$, it follows that $h_{\text{top}}(\mathsf{SwS}; \sigma, X_0) \leq R$. Since σ and R were arbitrary, this shows that $h_{\text{wc-top}}(\mathsf{SwS}) \leq \log_2(e) \hat{\varrho}(\mathsf{SwS}^\wedge)$, concluding the proof of Part 1. $\square$

Part 2: $h_{\text{wc-top}}(\mathsf{SwS}) \geq \log_2(e)\hat{\varrho}(\mathsf{SwS}^\wedge)$

Let $R < \log_2(e)\hat{\varrho}(\mathsf{SwS}^\wedge)$. By Proposition 1.52 in Subsection 1.3.2, there is $\sigma \in \mathcal{S}$ and $T \in \mathbb{T}_{>0}$ such that $\rho(\mathring{\chi}(T, 0, \sigma; \mathsf{SwS}^\wedge)) \geq 2^{RT}$, where $\rho(\mathring{\chi}(T, 0, \sigma; \mathsf{SwS}^\wedge))$ is the spectral radius of $\mathring{\chi}(T, 0, \sigma; \mathsf{SwS}^\wedge)$. Let $\bar{A} = \mathring{\chi}(T, 0, \sigma; \mathsf{SwS})$. By Proposition 3.22, it holds that $\mathring{\chi}(T, 0, \sigma; \mathsf{SwS}^\wedge) = \bar{A}^\wedge$, so that $\rho(\bar{A}^\wedge) \geq 2^{RT}$. Hence, by Item 3 in Proposition 3.20, the eigenvalues $\lambda_1(\bar{A}), \ldots, \lambda_n(\bar{A})$ of $\bar{A}$ satisfy that $\prod_{j=1}^n \max\{|\lambda_j(\bar{A})|, 1\} \geq 2^{RT}$. Hence, by (3.5), the discrete-time LTI system defined by $\bar{A}$ satisfies that $h_{\text{top}}(\bar{A}) = \log_2\left(\prod_{j=1}^n \max\{|\lambda_j(\bar{A})|, 1\}\right) \geq RT$.

Now, let $\sigma' : \mathbb{T}_{\geq 0} \to \Sigma$ be the switching signal defined by $\sigma'(t) = \sigma(t - \lfloor t/T \rfloor T)$ for all $t \in \mathbb{T}_{\geq 0}$, i.e., σ' is the repetition of $\sigma|_{[0,T)\cap\mathbb{T}}$ with period T. It follows that for every $k \in \mathbb{N}$, $\mathring{\chi}(kT, 0, \sigma') = \bar{A}^k$. Thus, the topological entropy of SwS with switching signal σ' satisfies $h_{\text{top}}(\mathsf{SwS}; \sigma') \geq h_{\text{top}}(\bar{A})/T \geq R$. Since $\sigma' \in \mathcal{S}$, it follows that $h_{\text{wc-top}}(\mathsf{SwS}) \geq R$. Since R was arbitrary, this shows that $h_{\text{wc-top}}(\mathsf{SwS}) \geq \log_2(e)\hat{\varrho}(\mathsf{SwS}^\wedge)$, concluding the proof of Part 2. □

A.3.6 Proof of Corollary 3.25

We prove the corollary for the discrete-time case, as the continuous-time case is identical. From Item 1 in Proposition 3.20, it holds that for each $i \in \Sigma$, $A_i^\wedge$ is normal. Hence, by classical results on the joint spectral radius (see, e.g., Jungers, 2009, Proposition 2.2), $\hat{\varrho}(\mathsf{SwS}^\wedge) = \max_{i\in\Sigma} \log(\rho(A_i^\wedge))$, where for each $i \in \Sigma$, $\rho(A_i^\wedge)$ is the spectral radius of $A_i^\wedge$. Hence, we get the conclusion of the corollary by using Item 3 in Proposition 3.20. The proof for switched linear systems with upper-/lower-triangular matrices is identical (see also Jungers, 2009, Proposition 2.3). □

A.3.7 Counter-example for Remark 3.3

Consider the discrete-time switched linear system system $\mathsf{SwS} \sim (\mathbb{R}^2, \{A_i\}_{i\in\Sigma})$, with $\Sigma = \{1, 2\}$, and $A_1 = \left[\begin{smallmatrix} 1 & \\ & 2 \end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix} 4 & \\ & 1/2 \end{smallmatrix}\right]$. Let $s : \mathbb{N} \to \mathbb{N}_{>0}$ be defined by $s(r) = 2^{r^2}$, and for each $m \in \mathbb{N}$, let $I(m) = \sum_{r=0}^{m-1} 2s(r)$. Define the switching signal $\sigma : \mathbb{N} \to \Sigma$ as follows: for all $t \in \mathbb{N}$, $\sigma(t) = 1$ if $I(m) \leq t < I(m) + s(m)$ for some $m \in \mathbb{N}$, and $\sigma(t) = 2$ if $I(m) + s(m) \leq t < I(m+1)$ for some $m \in \mathbb{N}$. We will show that $h_{\text{top}}(\mathsf{SwS}; \sigma) \neq \limsup_{T\to\infty} \frac{1}{T} \log_2 \|\mathring{\chi}(T, 0, \sigma)^\wedge\|$.

Norm of the exterior powers: We claim that $\limsup_{T\to\infty} \frac{1}{T} \log_2 \|\mathring{\chi}(T, 0, \sigma)^\wedge\| = 1$. To see this, we use that for any $A \in \mathbb{R}^{n\times n}$, $\|A^\wedge\| = \max_{k\in\{0,\ldots,n\}} \|A^{\wedge k}\|$ (by definition of $A^\wedge$). First, we consider the case $k = 0$. For all $T \in \mathbb{N}$, it holds that $\mathring{\chi}(T, 0, \sigma)^{\wedge 0} = 1$ (see Definition 3.18), whence $\|\mathring{\chi}(T, 0, \sigma)^{\wedge 0}\| = 1$. Hence, it follows that $\limsup_{T\to\infty} \frac{1}{T} \log_2 \|\mathring{\chi}(T, 0, \sigma)^{\wedge 0}\| = 0 \leq 1$.

Then, we consider the case $k = 2$. By Remark 3.1, it holds that for every $T \in \mathbb{N}$, $\|\mathring{\chi}(T,0,\sigma)^{\wedge 2}\| = |\det(\mathring{\chi}(T,0,\sigma))|$. Since $\det(A_1) = \det(A_2) = 2$, it follows that for every $T \in \mathbb{N}$, $\det(\mathring{\chi}(T,0,\sigma)) = 2^T$, so that $\limsup_{T\to\infty} \frac{1}{T} \log_2 \|\mathring{\chi}(T,0,\sigma)^{\wedge 2}\| = 1$.

Finally, we consider the case $k = 1$. Since A_1 and A_2 are diagonal, we may analyze the two components separately since $\limsup_{T\to\infty} \frac{1}{T} \log_2 \|\mathring{\chi}(T,0,\sigma)^{\wedge 1}\| = \max_{j\in\{1,2\}} \limsup_{T\to\infty} \frac{1}{T} \log_2 \bar{a}_j(T)$, where for each $T \in \mathbb{N}$ and $j \in \{1,2\}$, $\bar{a}_j(T)$ is the j^{th} diagonal entry of $\mathring{\chi}(T,0,\sigma)$. Let us analyze $\bar{a}_1$ first. It is not difficult to see that $T^{-1}\log_2 \bar{a}_1(T)$ is maximal when $T = I(m)$ for some $m \in \mathbb{N}$, and that the maximal value of $T^{-1}\log_2 \bar{a}_1(T)$ is equal to $2^{-1}\log_2 4 = 1$. As for the analysis of $\bar{a}_2$, observe that for all $T \in \mathbb{N}$, $\bar{a}_2(T) \leq 2^T$, so that $T^{-1}\log_2 \bar{a}_2(T) \leq 1$. This shows that $\limsup_{T\to\infty} \frac{1}{T} \log_2 \|\mathring{\chi}(T,0,\sigma)^{\wedge 1}\| = 1$. Putting things together, this proves that $\limsup_{T\to\infty} \frac{1}{T} \log_2 \|\mathring{\chi}(T,0,\sigma)^{\wedge}\| = 1$.

Lower bound on the entropy: We show that $h_{\text{top}}(\mathsf{SwS};\sigma) \geq 3/2$. Therefore, let $X_0 = [0,1]\times[0,1]$ and fix $\epsilon \in (0,1)$. Let $m \in \mathbb{N}$ and define $T_m = I(m+1)+1$. We build an (ϵ, T_m)-separated set $\mathcal{F}$ for SwS with switching signal σ and initial set X_0, such that $|\mathcal{F}| \geq 8^{s(m)}$, as follows: for each $p \in \{0,\ldots,4^{s(m)}\}$ and $q \in \{0,\ldots,2^{s(m)}\}$, let $x_{pq} = \left[\, p4^{-s(m)} \;\; q2^{-s(m)} \,\right]^\top \in X_0$, and let

$$\mathcal{F} = \{\, \chi(\cdot,0,x_{pq},\sigma)|_{\{0,\ldots,T_m-1\}} \,\}_{p\in\{0,\ldots,4^{s(m)}\},\, q\in\{0,\ldots,2^{s(m)}\}}.$$

We show that $\mathcal{F}$ is (ϵ, T_m)-separated for SwS with switching signal σ. Therefore, let $(p_1,q_1) \neq (p_2,q_2)$ and we show show that there is $t \in \{0,\ldots,T_m-1\}$ such that $\|\chi(t,0,x_{p_1q_1}) - \chi(t,0,x_{p_2q_2})\| > \epsilon$. First, assume that $p_1 \neq p_2$. Then, $|x^{(1)}_{p_1q_1} - x^{(1)}_{p_2q_2}| \geq 4^{-s(m)}$. Hence, we have that

$$|\chi^{(1)}(T_m-1,0,x_{p_1q_1}) - \chi^{(1)}(T_m-1,0,x_{p_2q_2})| = 4^{I(m+1)/2}|x^{(1)}_{p_1q_1} - x^{(1)}_{p_2q_2}| \geq 4^{s(m)}4^{-s(m)} > \epsilon.$$

Now, assume $q_1 \neq q_2$. Then, $|x^{(2)}_{p_1q_1} - x^{(2)}_{p_2q_2}| \geq 2^{-s(m)}$. Hence, we have that

$$\begin{aligned}|\chi^{(2)}(T_m - s(m) - 1, 0, x_{p_1q_1}) - \chi^{(2)}(T_m - s(m) - 1, 0, x_{p_2q_2})| &= \\ 2^{s(m)}|x^{(2)}_{p_1q_1} - x^{(2)}_{p_2q_2}| \geq 2^{s(m)}2^{-s(m)} &> \epsilon.\end{aligned}$$

This shows that $\mathcal{F}$ is (ϵ, T_m)-separated for SwS with switching signal σ. Also, it is clear that $|\mathcal{F}| > 8^{s(m)}$. Since m was arbitrary, it follows that for all $m \in \mathbb{N}$, $s_{\text{sep}}(\epsilon, I(m+1)+1; \mathsf{SwS}, \sigma, X_0) \geq 8^{s(m)}$. Finally, by definition of I, it holds that for all $m \in \mathbb{N}$, $2s(m) \leq I(m+1) \leq 2s(m) + 2ms(m-1)$. The definition of s then implies that $\lim_{m\to\infty}(I(m+1)+1)/s(m) = 2$. Hence, by injecting in (3.2), it follows that $h_{\text{top}}(\mathsf{SwS};\sigma) \geq \frac{1}{2}\log_2 8 = 3/2$.

Thus, we have shown that $h_{\text{top}}(\mathsf{SwS};\sigma) \geq 3/2 > 1 = \limsup_{T\to\infty} \frac{1}{T} \log_2 \|\mathring{\chi}(T,0,\sigma)^{\wedge}\|$, concluding the counter-example. □

A.3.8 Proof of $\mathcal{R}_{\text{stab-md}}(\mathsf{SwS}) \geq h_{\text{wc-top}}(\mathsf{SwS}^\circ)$ in Theorem 3.28

Let $R < h_{\text{wc-top}}(\mathsf{SwS}^\circ)$. Fix $\epsilon > 0$ and let $\mathbb{B}$ be the centered unit Euclidean ball in $\mathbb{R}^n$. Assume that SwS is stabilizable with a mode-dependent coder–decoder CoDec with data rate smaller than or equal to R. Then, there is $T_* \in \mathbb{T}_{\geq 0}$ such that for all $x \in \mathbb{B}$, $\sigma \in \mathcal{S}$ and $t \in \mathbb{T}$, $t \geq T_*$, it holds that $\|\chi(t, 0, x, \sigma; \mathsf{SwS}\|\mathsf{CoDec})\| \leq \epsilon/2$, where $\mathsf{SwS}\|\mathsf{CoDec}$ is the closed-loop system obtained from the feedback composition of SwS and CoDec.

On the other hand, by definition of the worst-case topological entropy, there is $\sigma \in \mathcal{S}$ such that $h_{\text{top}}(\mathsf{SwS}; \sigma^\circ) > R$. Hence, by Proposition 1.79 in Subsection 1.5.2, there is $T \in \mathbb{T}$, $T \geq T_*$, and a set $F \subseteq \mathbb{B}$ such that $|F| > 2^{R(T+1)}$ and for any distinct $x_1, x_2 \in F$, there is $t \in [T_*, T) \cap \mathbb{T}$ satisfying $\|\chi(t, 0, x_1, \sigma; \mathsf{SwS}^\circ) - \chi(t, 0, x_2, \sigma; \mathsf{SwS}^\circ)\| > \epsilon$.

By the assumption on its data rate, CoDec cannot output, for a fixed switching signal, more than $2^{R(T+1)}$ different control inputs during the interval $[0, T) \cap \mathbb{T}$. This implies that there are at least two distinct $x_1, x_2 \in F$ for which the control input produced by CoDec will be the same for the trajectories starting from x_1 and x_2, and with switching signal σ. Denote this control input by u. By definition of T_*, it follows that for all $t \in [T_*, T) \cap \mathbb{T}$ and $j \in \{1, 2\}$, $\|\chi(t, 0, x_j, \sigma, u; \mathsf{SwS})\| \leq \epsilon/2$. Also, by the linearity of the system, it holds that for all $t \in [0, T) \cap \mathbb{T}$, $\chi(t, 0, x_1, \sigma, u; \mathsf{SwS}) - \chi(t, 0, x_2, \sigma, u; \mathsf{SwS}) = \chi(t, 0, x_1, \sigma; \mathsf{SwS}^\circ) - \chi(t, 0, x_2, \sigma; \mathsf{SwS}^\circ)$. It follows that $\|\chi(t, 0, x_1, \sigma; \mathsf{SwS}^\circ) - \chi(t, 0, x_2, \sigma; \mathsf{SwS}^\circ)\| \leq \epsilon$. This is a contradiction with the definition of F, which implies that $\|\chi(t, 0, x_1, \sigma; \mathsf{SwS}^\circ) - \chi(t, 0, x_2, \sigma; \mathsf{SwS}^\circ)\| > \epsilon$. Thus, there cannot exist a mode-dependent coder–decoder, with data rate smaller than or equal to R, that stabilizes SwS. Since R was arbitrary, this shows that $\mathcal{R}_{\text{stab-md}}(\mathsf{SwS}) \geq h_{\text{wc-top}}(\mathsf{SwS}^\circ)$, which concludes the proof. □

A.3.9 Proof of Lemma 3.31

First, we derive, for any $A \in \mathbb{R}^{n\times n}$, an upper bound on $\hat{m}(\alpha, A)$ as a function of $A^\wedge$. Therefore, fix $A \in \mathbb{R}^{n\times n}$ and let $c = n^{1/2}/(2\alpha)$. Note that for any $r \in \mathbb{R}$, it holds that $[\![r]\!] \leq r + \frac{1}{2}$. Hence, by Item 2 in Lemma 3.30, $\hat{m}(\alpha, A) \leq \prod_{i=1}^n (2c + 2) \max\{\rho_i, 1\}$, where $\rho_1, \ldots, \rho_n$ are the singular values of A. It follows, by Item 4 in Proposition 3.20, that $\hat{m}(\alpha, A) \leq (2c + 2)^n \|A^\wedge\|$.

Now, using the above result, we prove the lemma. Therefore, let $\lambda \in \mathbb{R}$ be such that $\hat{\varrho}(\mathsf{SwS}^\wedge) < \lambda < R/\log_2(e)$. Then, by the definition of the joint spectral radius (see Definition 1.50 in Subsection 1.3.2), there is $C \geq 0$ such that for all $\sigma \in \mathcal{S}$ and $T \in \mathbb{T}_{\geq 0}$, $\|\mathring{\chi}(T, 0, \sigma)^\wedge\| \leq Ce^{\lambda T}$. Thus, there is $T \in \mathbb{T}_{\geq 0}$

such that for all $\sigma \in \mathcal{S}$ and $T' \in \mathbb{T}$, $T' \geq T$, it holds that $\hat{m}(\alpha, \mathring{\chi}(T', 0, \sigma)) \leq (2c+2)^n \|\mathring{\chi}(T', 0, \sigma)^\wedge\| \leq 2^{\lfloor RT' \rfloor}$, which concludes the proof. $\square$

A.3.10 Proof of the correctness of the coder–decoder in Figure 3.7

Let CoDec be the coder–decoder described in Figure 3.7, with the parameters defined in the paragraph "Parameters". First, we show that $\mathcal{R}(\mathsf{CoDec}) \leq R$. From the definition of the period T of CoDec, it holds that for every $k \in \mathbb{N}$, $\hat{m}(\alpha, \bar{A}_k) \leq 2^{\lfloor RT \rfloor}$. Hence, for each $k \in \mathbb{N}$, $\hat{y}_k$ can be encoded as a symbol $e(k)$ of at most $\lfloor RT \rfloor$ bits. It follows, by the definition of $\mathcal{R}(\mathsf{CoDec})$ (see Definition 1.73 in Subsection 1.5.1), that $\mathcal{R}(\mathsf{CoDec}) = \lfloor RT \rfloor / T \leq R$ bits per unit of time.

Now, we show that CoDec satisfies (3.6). Therefore, let $(\xi, \sigma) : \mathbb{T}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ be a trajectory of the closed-loop system $\mathsf{SwS}\|\mathsf{CoDec}$. First, we show by induction on k that for every $k \in \mathbb{N}$, it holds that $\|\xi(kT) - \hat{x}_k\| \leq \alpha^k$. Indeed, this holds trivially for $k = 0$, since $\xi(0) \in \mathbb{B}$. Now, assume that it is true for some $k \in \mathbb{N}$, and observe that, by definition of y_{k+1} and $\bar{A}_{k+1}$ and by the linearity of the system, $y_{k+1} = \bar{A}_{k+1}(\xi(kT) - \hat{x}_k)$, and thus, by the induction hypothesis, $y_{k+1} \in \alpha^k \bar{A}_{k+1}\mathbb{B}$. Hence, by definition of $\hat{y}_{k+1}$, it holds that $\|\hat{y}_{k+1} - y_{k+1}/\alpha^k\| \leq \alpha$. By definition of $\hat{x}_{k+1}$, it then follows that $\|\xi((k+1)T) - \hat{x}_{k+1}\| \leq \alpha^{k+1}$, concluding the proof of the induction step.

Secondly, we show that $\hat{x}_k \to 0$ exponentially as $k \to \infty$. Therefore, let $L = \max_{i \in \Sigma} \|A_i\|$, and note that for all $k \in \mathbb{N}$, $\|\bar{A}_k\| \leq e^{LT}$ so that $\|\hat{y}_k\| \leq e^{LT} + \alpha$ (by definition of $\hat{y}_k$ and since $y_k \in \alpha^{k-1}\bar{A}_k\mathbb{B}$). Thus, it holds that for each $k \in \mathbb{N}$,

$$
\begin{aligned}
n_{k+1} \doteq \|\hat{x}_{k+1}\| &= \|n_k\, \chi((k+1)T, kT, \hat{x}_k/n_k, \sigma; \mathsf{SwS}\|\kappa) + \alpha^k \hat{y}_{k+1}\| \\
&\leq \|n_k\, \chi((k+1)T, kT, \hat{x}_k/n_k, \sigma; \mathsf{SwS}\|\kappa)\| + \alpha^k (e^{LT} + \alpha) \\
&\leq \alpha n_k + \alpha^k (e^{LT} + \alpha),
\end{aligned}
$$

where the last inequality follows from the definition of T. It follows that there is $C \geq 0$ and $\theta \in (\alpha, 1)$ such that for all $k \in \mathbb{N}$, $n_k \leq C\theta^k$, showing that $\hat{x}_k \to 0$ exponentially as $k \to \infty$.

Finally, using the above results, we show that $\xi(t) \to 0$ exponentially as $t \to \infty$. To see this, note that since κ is a stabilizing controller, there is $M \geq 0$ such that for all $x \in \mathbb{B}$, $\sigma \in \mathcal{S}$ and $t \in \mathbb{T}_{\geq 0}$, it holds that $\|\chi(t, 0, x, \sigma; \mathsf{SwS}\|\kappa)\| \leq M$.

Hence, it follows that for all $k \in \mathbb{N}$ and $t \in [kT, (k+1)T) \cap \mathbb{T}$,

$$\begin{aligned}\|\xi(t)\| &= \|n_k\,\chi(t, kT, \hat{x}_k/n_k, \sigma; \mathsf{SwS}\|\kappa) + \mathring{\chi}(t, kT, \sigma)(\xi(kT) - \hat{x}_k)\| \\ &\leq Mn_k + e^{L(t-kT)}\alpha^k \leq CM\theta^k + e^{LT}\alpha^k \leq \frac{1}{\theta}(CM + e^{LT})\theta^{k+1} \leq C'e^{-\gamma t},\end{aligned} \tag{A.21}$$

where $C' = \frac{1}{\theta}(CM + e^{LT})$ and $\gamma = -\frac{1}{T}\log\theta > 0$.

To prove that CoDec satisfies (3.6), it remains to show that the closed-loop system $\mathsf{SwS}\|\mathsf{CoDec}$ is *Lyapunov stable*, meaning that there is a class-$\mathcal{K}$ function h such that every trajectory $(\xi, \sigma) : \mathbb{T}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ of $\mathsf{SwS}\|\mathsf{CoDec}$ satisfies $\|\xi(t)\| \leq h(\|\xi(0)\|)$ for all $t \in \mathbb{T}_{\geq 0}$. The proof of this claim is along the same lines as the proof of Liberzon and Hespanha (2005, Theorem 1), and thus omitted here.[19]

Finally, we combine the above Lyapunov stability property with the exponential decay property (A.21), to show that the CoDec satisfies (3.6). Therefore, let $\mu \in (0, \gamma)$. It is readily seen that for all $t \in \mathbb{T}_{\geq 0}$, $\|\xi(t)\| = \|\xi(t)\|^{1-\mu/\gamma}\|\xi(t)\|^{\mu/\gamma} \leq h(\|\xi(0)\|)^{1-\mu/\gamma}(C'e^{-\gamma t})^{\mu/\gamma}$. Hence, we get the desired property, by letting $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be defined by $g(r) = h(r)^{1-\mu/\gamma}C'^{\mu/\gamma}$ which is clearly a class-$\mathcal{K}$ function. This concludes the proof that CoDec satisfies (3.6). □

A.3.11 Proof of Theorem 3.35

We present a sketch of proof with the help of an example. Indeed, a complete proof of the theorem would require many technical and cumbersome developments, while we believe that the main ideas of the proof can be deduced from the example.

Example A.9. Consider the switched linear system $\mathsf{SwS} \sim (\mathbb{R}^1, \{A_i\}_{i\in\Sigma})$ with $\Sigma = \{1, 2\}$, and $A_1 = 0$ and $A_2 = 1$. Let $\tau_{\mathrm{a}} > 0$ and assume that $\mathcal{S}$ contains all switching signals from $\mathbb{R}_{\geq 0}$ to Σ with absolute dwell time τ_{a}. Let $X_0 = [0, 1]$. We will show that $h_{\mathrm{top}}(\mathsf{SwS}, X_0) = \infty$.

Therefore, let $R > 0$ and let $S \subseteq [1, 2]$ be a finite set with cardinality $|S| \geq 2^{3R\tau_{\mathrm{a}}}$. Let $\epsilon > 0$ be such that for any distinct $s_1, s_2 \in S$, it holds that $|e^{(3-s_1)\tau_{\mathrm{a}}} - e^{(3-s_2)\tau_{\mathrm{a}}}| > \epsilon$. We show that for all $k \in \mathbb{N}$, $s_{\mathrm{sep}}(\epsilon, 3k\tau_{\mathrm{a}}; X_0) \geq 2^{3Rk\tau_{\mathrm{a}}}$.

To show this, fix $k \in \mathbb{N}$, and for each sequence $\bar{v} \doteq (v_j)_{j=0}^{k-1} \in S^k$, let $\sigma_{\bar{v}} : \mathbb{R}_{\geq 0} \to \Sigma$ be defined as follows: for every $t \in \mathbb{R}_{\geq 0}$, $\sigma_{\bar{v}}(t) = 1$ if $3j\tau_{\mathrm{a}} \leq t < (3j+$

[19] In fact, Liberzon and Hespanha (2005, Theorem 1) shows Lyapunov stability in terms of the "ε–δ definition". The equivalence of the "ε–δ definition" with the "class-$\mathcal{K}$ function definition" can be found, e.g., in Khalil (2002, Lemma 4.5), or Clarke et al. (1998, Lemma 2.5).

$v_j)\tau_a$ for some $j \in \{0,\dots,k-1\}$, and $\sigma_{\bar{v}}(t) = 2$ otherwise. It is clear that, for each $\bar{v} \in S^k$, $\sigma_{\bar{v}}$ has absolute dwell time τ_a. Also, for any $\bar{v} \doteq (v_j)_{j=0}^{k-1} \in S^k$, $x \in \mathbb{R}^1$ and $j \in \{0,\dots,k-1\}$, it holds that $\chi(3(j+1)\tau_a, 3j\tau_a, x, \sigma_{\bar{v}}) = e^{(3-v_j)\tau_a}x$. Let $\mathcal{F} = \{\phi_{\bar{v}}|_{[0,3k\tau_a)}\}_{\bar{v}\in S^k}$, where for each $\bar{v} \in S^k$, $\phi_{\bar{v}} = (\xi_{\bar{v}}, \sigma_{\bar{v}}) : \mathbb{R}_{\geq 0} \to \mathbb{R}^1 \times \Sigma$ is the trajectory of SwS with switching signal $\sigma_{\bar{v}}$ and with $\xi_{\bar{v}}(0) = 1$. We show that $\mathcal{F}$ is $(\epsilon, 3k\tau_a)$-separated for SwS starting from X_0.

Therefore, let $\bar{v}_1 \doteq (v_{1,j})_{j\in\mathbb{N}} \in S^k$ and $\bar{v}_2 \doteq (v_{2,j})_{j\in\mathbb{N}} \in S^k$ be such that $\bar{v}_1 \neq \bar{v}_2$ and let $j \in \{0,\dots,k-1\}$ be the smallest index such that $v_{1,j} \neq v_{2,j}$. Then, it holds that $\xi_{\bar{v}_1}(t) = \xi_{\bar{v}_2}(t)$ for all $t \in [0, 3j\tau_a]$. However, $\xi_{\bar{v}_1}(3(j+1)\tau_a) = e^{(3-v_{1,j})\tau_a}\xi_{\bar{v}_1}(3j\tau_a) \neq \xi_{\bar{v}_2}(3(j+1)\tau_a) = e^{(3-v_{2,j})\tau_a}\xi_{\bar{v}_2}(3j\tau_a)$. Since for all $t \in \mathbb{R}_{\geq 0}$, $\xi_{\bar{v}_1}(t) \geq \xi_{\bar{v}_1}(0) = 1$ and $\xi_{\bar{v}_2}(t) \geq \xi_{\bar{v}_2}(0) = 1$, it follows, by definition of ϵ, that $|\xi_{\bar{v}_1}(3(j+1)\tau_a) - \xi_{\bar{v}_2}(3(j+1)\tau_a)| > \epsilon$, which implies that $\mathcal{F}$ is $(\epsilon, 3k\tau_a)$-separated for SwS starting from X_0.

Finally, the cardinality of $\mathcal{F}$ is equal to $|S|^k$, so that $s_{\mathrm{sep}}(\epsilon, 3k\tau_a; X_0) \geq |S|^k \geq 2^{3Rk\tau_a}$. Since k was arbitrary, this shows that for every $k \in \mathbb{N}$, $s_{\mathrm{sep}}(\epsilon, 3k\tau_a; X_0) \geq |S|^k \geq 2^{3Rk\tau_a}$. Hence, by (3.8), it follows that $h_{\mathrm{top}}(\mathsf{SwS}, X_0) \geq R$. Since R was arbitrary, this shows that $h_{\mathrm{top}}(\mathsf{SwS}, X_0) = \infty$, concluding the example. □

A.3.12 Proof of the correctness of the coder–decoder in Figure 3.11

Let CoDec be the coder–decoder described in Figure 3.11, with the parameters defined in the paragraph "Parameters". First, we show that $\mathcal{R}(\mathsf{CoDec})$ satisfies (3.12). For each $k \in \mathbb{N}$, it holds that $(\hat{y}_k, v_k) \in \Xi_\alpha \times \{0,\dots,p\}$. Hence, for each $k \in \mathbb{N}$, $(\hat{y}_k, v_k)$ can be encoded with $\lceil \log_2(|\Xi_\alpha|(p+1)) \rceil$ bits, which implies that $\lceil \frac{1}{p} \lceil \log_2(|\Xi_\alpha|(p+1)) \rceil \rceil$ bits per symbol $e(kp+j)$, $j \in \{0,\dots,p-1\}$, is sufficient to encode $(\hat{y}_k, v_k)$. On the other hand, for each $j \in \mathbb{N}$, $\sigma((kp+j)T)$ can be encoded with $\lceil \log_2(|\Sigma|) \rceil$ bits. Hence, the coder needs to send at most $\lceil \frac{1}{p} \lceil \log_2(|\Xi_\alpha|(p+1)) \rceil \rceil + \lceil \log_2(|\Sigma|) \rceil$ bits at each time $t = (kp+j)T$, $k \in \mathbb{N}$ and $j \in \{0,\dots,p-1\}$. This shows that $\mathcal{R}(\mathsf{CoDec})$ satisfies (3.12).

Now, we show that CoDec satisfies (3.10). Therefore, let $(\xi, \sigma) : \mathbb{T}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ be a trajectory of the closed-loop system $\mathsf{SwS}\|\mathsf{CoDec}$. First, we show by induction on k that for all $k \in \mathbb{N}$, it holds that $\|\xi(kpT)\| \leq r_k$, $\|\hat{x}_k\| \leq r_k$ and $\|\xi(kpT) - \hat{x}_k\| \leq s_k$. Indeed, this holds trivially true for $k = 0$. Now, assume that it it true for some $k \in \mathbb{N}$. Then, by Eq. (11) in Berger and Jungers (2020b) and by definition of $\bar{b}$, $\tilde{x}_{k+1}$ and v_k, it follows that

$$\|\xi((k+1)pT) - \tilde{x}_{k+1}\| \leq e^{\nu pT} s_k + \bar{b}(v_k) r_k. \tag{A.22}$$

Also, by assumption on κ and by definition of $\bar{a}$ and v_k, it holds that

$$\|\tilde{x}_{k+1}\| \leq \bar{a}(v_k)\|\hat{x}_k\| \leq \bar{a}(v_k)r_k. \tag{A.23}$$

Putting things together and by definition of r_{k+1}, this shows that $\|\xi((k+1)pT)\| \leq r_{k+1}$. Furthermore, by definition of y_k and $\hat{y}_k$, it holds that $\|\xi(kpT)-\hat{x}_k - s_k\hat{y}_k\| \leq \alpha s_k$. Then, by a reasoning similar as above, it follows that $\|\xi((k+1)pT) - \hat{x}_{k+1}\| \leq e^{\nu pT}\alpha s_k + \bar{b}(v_k)r_k = s_{k+1}$ and $\|\hat{x}_{k+1}\| \leq \bar{a}(v_k)\|\hat{x}_k\| \leq r_{k+1}$, concluding the proof of the induction step.

Now, we show that $\xi(kpT) \to 0$ exponentially as $k \to \infty$. By the above result, it suffices to show that $r_k \to 0$ exponentially as $k \to \infty$. Therefore, for each $k \in \mathbb{N}$, let $\omega_k = \max\{r_k, \frac{1}{\alpha}s_k\}$. Then, by definition of r_k and s_k, it holds that for all $k \in \mathbb{N}$, $\omega_{k+1} \leq \beta_k\omega_k$, where

$$\beta_k = \bar{a}(v_{k+1}) + (1+\tfrac{1}{\alpha})\bar{b}(v_{k+1}) + e^{\nu pT}\alpha.$$

Thus, for all $k \in \mathbb{N}$, $\omega_k \leq \omega_0\prod_{j=0}^{k-1}\beta_j$. We show that $\limsup_{k\to\infty}(\prod_{j=0}^{k-1}\beta_j)^{1/k} < \theta < 1$. Therefore, fix $k \in \mathbb{N}$, and note that

$$\left(\prod_{j=0}^{k-1}\beta_j\right)^{1/k} \leq e^{\mu_1\frac{1}{k}\sum_{j=0}^{k-1}v_{j+1}}\prod_{j=0}^{k-1}\big(De^{-\mu_2 pT}+(1+\tfrac{1}{\alpha})e^{\nu pT}v_{j+1}TD(\Delta_A+\Delta_B L)+e^{\nu pT}\alpha\big)^{1/k}.$$

From the arithmetic–geometric mean inequality, it follows that

$$\left(\prod_{j=0}^{k-1}\beta_j\right)^{1/k} \leq e^{\mu_1\frac{1}{k}\sum_{j=0}^{k-1}v_{j+1}}\frac{1}{k}\sum_{j=0}^{k-1}\big(De^{-\mu_2 pT}+(1+\tfrac{1}{\alpha})e^{\nu pT}v_{j+1}TD(\Delta_A+\Delta_B L)+e^{\nu pT}\alpha\big),$$

and since for all $j \in \mathbb{N}$, $v_{j+1} \leq N_\sigma((j+1)T, jT)$, we get that

$$\left(\prod_{j=0}^{k-1}\beta_j\right)^{1/k} \leq e^{\mu_1\frac{1}{k}N_\sigma(kT,0)}\big(De^{-\mu_2 pT}+(1+\tfrac{1}{\alpha})e^{\nu pT}\tfrac{1}{k}N_\sigma(kT,0)TD(\Delta_A+\Delta_B L)+e^{\nu pT}\alpha\big),$$

Since k was arbitrary, the above holds for every $t \in \mathbb{N}$. Now, by assumption on σ having average dwell time τ_{a} and definition of T, α and p satisfying (3.11), it follows that $\limsup_{k\to\infty}(\prod_{j=0}^{k-1}\beta_j)^{1/k} < \theta$. Thus, there is $C \geq 0$ such that for all $k \in \mathbb{N}$, $r_k \leq C\theta^k$, showing that $\xi(kpT) \to 0$ exponentially as $k \to \infty$.

Finally, from the above, it follows that $\xi(t) \to 0$ exponentially as $t \to \infty$. Indeed, by using an argument similar to the one used in (A.22)–(A.23) (or simply by using the continuity of the trajectories of switched systems, but this was not proved formally), it follows that there is $M \geq 0$ such that for all $k \in \mathbb{N}$ and $t \in [kpT, (k+1)pT)$, $\|\xi(t)\| \leq M\|\xi(kpT)\|$. Hence, for all $t \in \mathbb{R}_{\geq 0}$, it holds that

$$\|\xi(t)\| \leq \frac{1}{\theta}MC\theta^{k+1} \leq C'e^{-\gamma t}, \tag{A.24}$$

where $C' = \frac{1}{\theta} MC$ and $\gamma = -\frac{1}{pT} \log \theta > 0$.

To prove that CoDec satisfies (3.10), it remains to show that the closed-loop system SwS‖CoDec is *Lyapunov stable*, meaning that there is a class-$\mathcal{K}$ function h such that every trajectory $(\xi, \sigma) : \mathbb{T}_{\geq 0} \to \mathbb{R}^n \times \Sigma$ of SwS‖CoDec satisfies $\|\xi(t)\| \leq h(\|\xi(0)\|)$ for all $t \in \mathbb{T}_{\geq 0}$. The proof of this claim is along the same lines as the proof of Liberzon and Hespanha (2005, Theorem 1), and thus omitted here.[20]

Finally, we combine the above Lyapunov stability property with the exponential decay property (A.24), to show that the CoDec satisfies (3.10). Therefore, let $\mu \in (0, \gamma)$. In other words, μ can be real satisfying

$$0 < \mu < -\frac{1}{pT} \log \theta. \tag{A.25}$$

It is readily seen that for all $t \in \mathbb{T}_{\geq 0}$, $\|\xi(t)\| = \|\xi(t)\|^{1-\mu/\gamma} \|\xi(t)\|^{\mu/\gamma} \leq h(\|\xi(0)\|)^{1-\mu/\gamma} (C' e^{-\gamma t})^{\mu/\gamma}$. Hence, we get the desired property, by letting $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be defined by $g(r) = h(r)^{1-\mu/\gamma} C'^{\mu/\gamma}$ which is clearly a class-$\mathcal{K}$ function. This concludes the proof that CoDec satisfies (3.10). □

[20] In fact, Liberzon and Hespanha (2005, Theorem 1) shows Lyapunov stability in terms of the "ε–δ definition". The equivalence of the "ε–δ definition" with the "class-$\mathcal{K}$ function definition" can be found, e.g., in Khalil (2002, Lemma 4.5), or Clarke et al. (1998, Lemma 2.5).

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