Formal methods for computing hyperbolic invariant sets of nonlinear systems

UCLouvain

Guillaume O. Berger[§] and Raphaël M. Jungers[§]

[§] UCLouvain, Belgium HSCC, April 2019, Montreal

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What & Why

Given

Dynamical system: $x_{t+1} = f(x_t)$

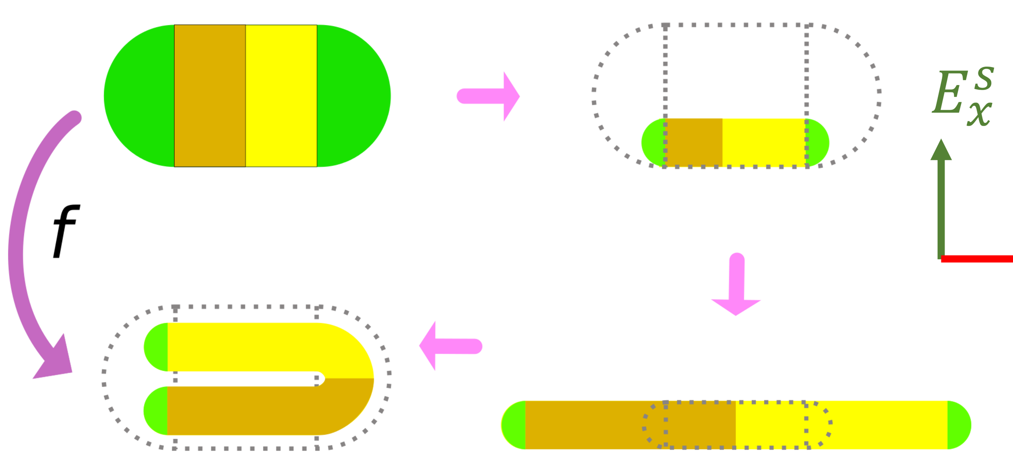
$f: M \to M$ – C^1 diffeomorphism

M: compact Riemannian manifold of dimension d

Objective
Compute a **hyperbolic** invariant set for f

($\Lambda \subseteq M$ invariant set $\Leftrightarrow f(\Lambda) = \Lambda$)

Definition (informal): The invariant set $\Lambda \subseteq M$ is hyperbolic if for every $x \in \Lambda$ there exists a splitting $T_x M = E_x^u \oplus E_x^s$ with $\dim(E_x^u) = p$ and $\dim(E_x^s) = d - p$ such that f is *infinitesimally expanding* along E_x^u (unstable direction) and *infinitesimally contracting* along E_x^s (stable direction).

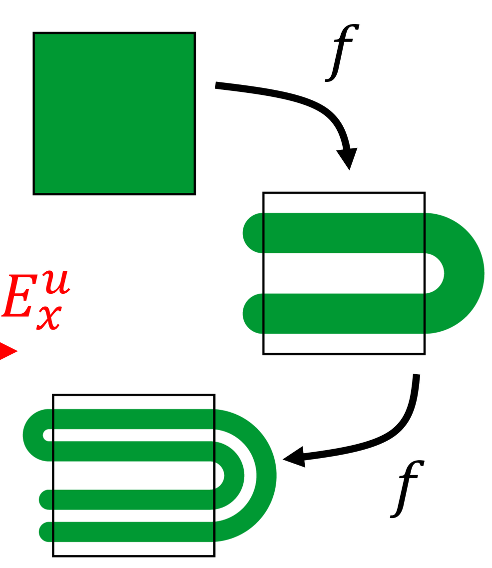


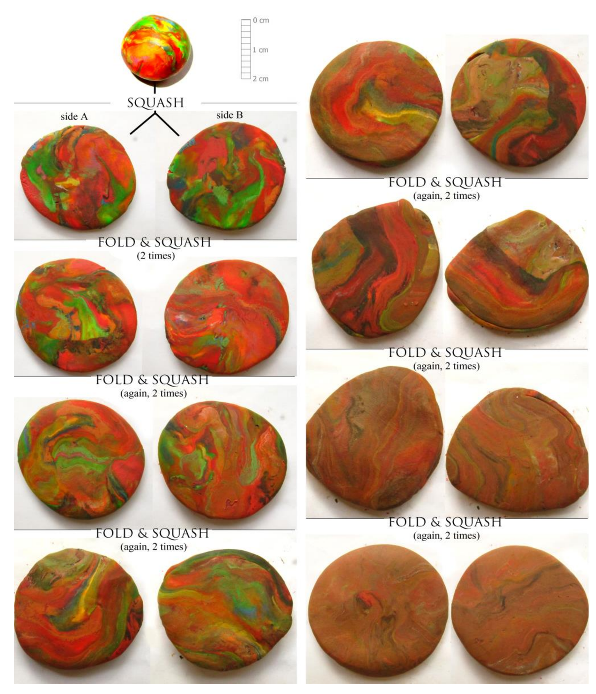
Applications

→ Structural stability: **robustness** to system perturbation and computation errors

→ Ergodic properties: e.g., mixing

application: e.g., statistical physics





How

Two main ingredients of our algorithmic framework: **Lyapunov-like criterion** and **abstraction of the system**

$$Df_x^T S_{f(x)} Df_x - S_x \prec 0 \qquad (*)$$

Df_x: Jacobian matrix of f

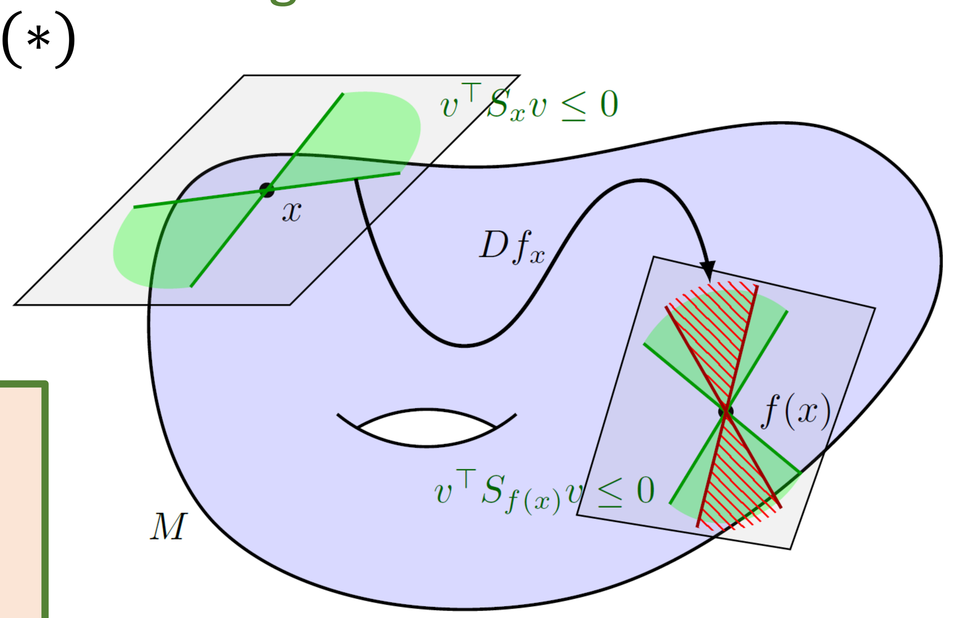
S_x: Symmetric matrix with p eigenvalues < 0 and $d - p$ eigenvalues > 0

Theorem: Given $\Lambda \subseteq M$ invariant

$\forall x \in \Lambda$, $\exists S_x$ satisfying $(*)$

$\Downarrow$

Λ is hyperbolic

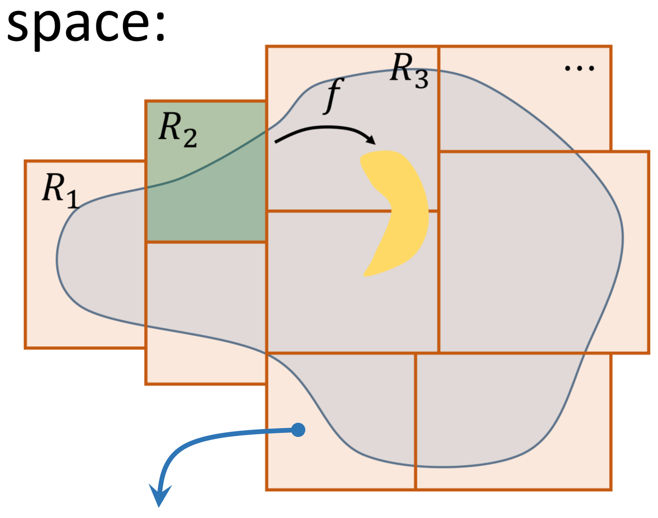


Advantage: $(*)$ is a LMI and thus can be checked with SDP solvers

Problem? There is an infinite number of symmetric matrices S_x to find…

Solution

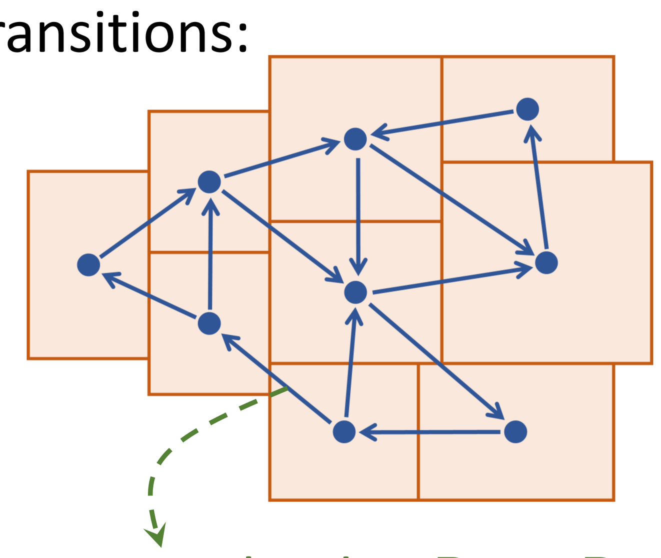
Discretization of the state space:



Assume S_x constant on each region: $S_x = \hat{S}_i$ on region R_i

+

Graph representing the transitions:



For each edge $R_i \to R_j$, let $A_{ij} \in \mathbb{R}^{d \times d}$

= Abstraction of the system

δ-approximation of Df
Assume $\|Df_x - A_{ij}\| \le \delta \|A_{ij}\|^{-1}$ for all $x \in R^i \cap f^{-1}(R_j)$

Algorithm: Given abstraction and A_{ij} as above, **find** $\hat{S}_i$ such that

$$(\star) \qquad A_{ij}^T \hat{S}_j A_{ij} - \hat{S}_i \preccurlyeq -\varepsilon I$$

for every edge $R_i \to R_j$ in the abstraction, and with $\varepsilon > 2\delta + \delta^2$

↑↑↑ ***Finite set of LMIs: Can be solved efficiently with SDP solvers***

Implementation

(Illustration on a numerical example)

The modified Ikeda map:

$$f(x, y) = (r + a(x\cos(\tau) - y\sin(\tau)), b(x\sin(\tau) + y\cos(\tau))$$

where $\tau = A - \frac{B}{1+x^2+y^2}$, $r = 2$, $A = 0.4$, $B = 6$, $a = -b = 0.9$

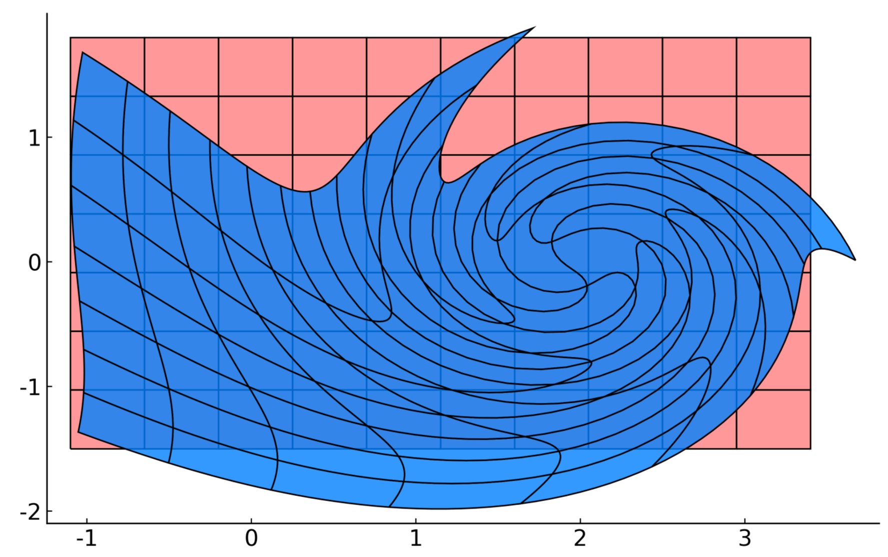
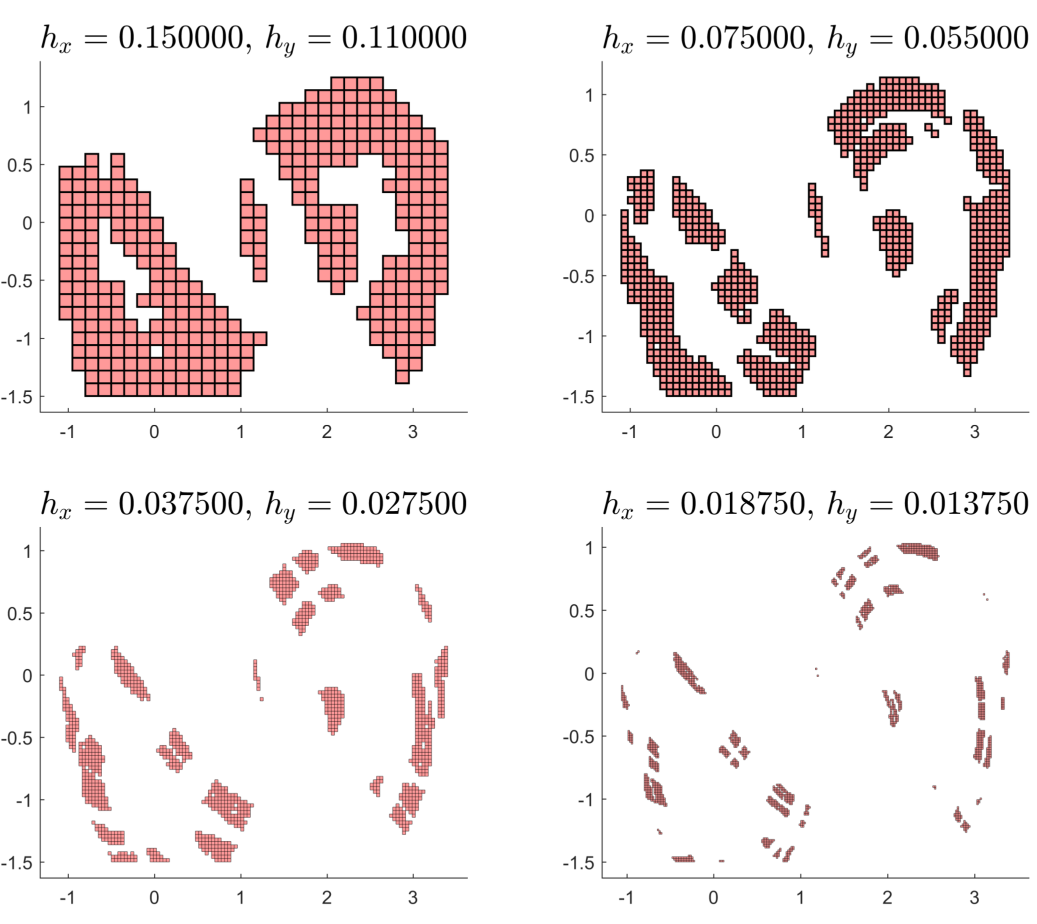


Image of the domain $M = [1.1, 3.4] \times [1.5, 1.8]$ by the Ikeda map f

Objective: find a hyperbolic invariant set in this domain

Step 1: Discretize the state space and build an abstraction of the system



Step 2: Find the recurrent regions (i.e., that are in a nontrivial cycle)

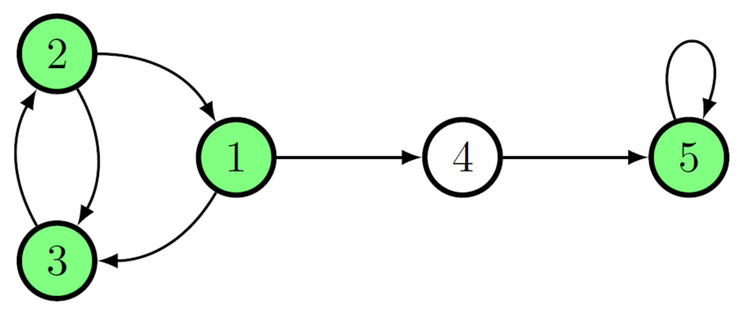


Fig. 2. Directed graph. The vertices 1, 2, 3, 5 are recurrent.

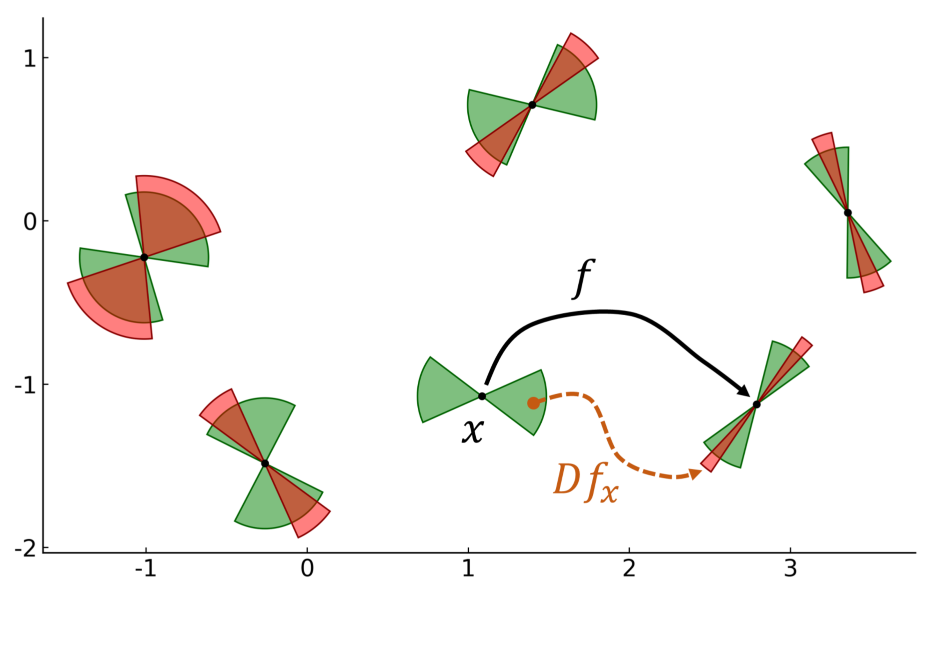
→ This gives an over-approximation of the largest invariant set in M

Refine the discretization to have a better over-approximation

Step 3: For a desired accuracy δ, find A_{ij} for each edge (i, j) such that it is a δ-approximation of Df

Refine the discretization if necessary…

Step 4: Solve the SDP feasibility problem $(\star)$



Output of solver: $\hat{S}_i$

Green cones: set of vectors v such that $v^T \hat{S}_i v \le 0$

Red cones: image of green cone by Df_x